Manifold Atlas: supplementary material for the page **Formal group laws and genera**

Theorem. *For every genus $\varphi\colon \Omega_U \to R\otimes\mathbb{Q}$, the exponential of the formal group law $\varphi(F_U)$ is given by the series $f(x)\in R\otimes\mathbb{Q}[[x]]$ corresponding to φ.*

This can be proved either directly, by appealing to the construction of geometric cobordisms, or indirectly, by calculating the values of the give Hirzebruch genus on projective spaces and comparing to the formula for the logarithm of the formal group law.

1st proof. Let X be a manifold and $u,v\in U^2(X)$ its two geometric cobordisms defined by the elements $x,y\in H^2(X)$ respectively. By the definition of the formal group law $F_U(u,v)=u+v+\sum_{k\geq 1,l\geq 1}\alpha_{kl}u^kv^l$ we have the following relation between geometric cobordisms in $U^2(X)$:

$$[M_{x+y}]=\sum_{k,l\geq 0}\alpha_{kl}[M_{x^ky^l}]$$

in Ω_U, where $M_{x+y}\subset X$ is the codimension 2 submanifold dual to $x+y\in H^2(X)$, and $M_{x^ky^l}\subset X$ is a codimension $2(k+l)$ submanifold dual to $x^ky^l\in H^2(X)$. Applying the genus φ we obtain

$$\varphi[M_{x+y}]=\sum\varphi(\alpha_{kl})\varphi[M_{x^ky^l}]. \tag{1}$$

Let $\iota\colon M_{x+y}\subset X$ be the embedding. Considering the decomposition

$$\iota^*(\mathcal{T}X)=\mathcal{T}M_{x+y}\oplus\nu(\iota)$$

and using the multiplicativity of the characteristic class φ we obtain

$$\iota^*\varphi(\mathcal{T}X)=\varphi(\mathcal{T}M_{x+y})\cdot\iota^*\big(\tfrac{x+y}{f(x+y)}\big).$$

Therefore,

$$\varphi[M_{x+y}]=\iota^*\big(\varphi(\mathcal{T}X)\cdot\tfrac{f(x+y)}{x+y}\big)\langle M_{x+y}\rangle=\big(\varphi(\mathcal{T}X)\cdot f(x+y)\big)\langle X\rangle. \tag{2}$$

Similarly, by considering the embedding $M_{x^ky^l}\to X$ we obtain

$$\varphi[M_{x^ky^l}]=\big(\varphi(\mathcal{T}X)\cdot f(x)^kf(y)^l\big)\langle X\rangle. \tag{3}$$

Plugging (2) and (3) into (1) we finally obtain

$$f(x+y)=\sum_{k,l\geq 0}\varphi(\alpha_{kl})f(x)^kf(y)^l.$$

This implies, by definition, that f is the exponential of $\varphi(F_U)$. □

2nd proof. The complex bundle isomorphism $\mathcal{T}(\mathbb{C}P^k)\oplus\underline{\mathbb{C}}=\bar\eta\oplus\ldots\oplus\bar\eta$ ($k+1$ summands) allows us to calculate the value of a genus on $\mathbb{C}P^k$ explicitly. Let $x=c_1(\bar\eta)\in H^2(\mathbb{C}P^k)$ and let g be the series functionally inverse to f; then

$$\begin{aligned}\varphi[\mathbb{C}P^k]&=\Big(\frac{x}{f(x)}\Big)^{k+1}\langle\mathbb{C}P^k\rangle\\&=\text{coefficient of } x^k \text{ in } \Big(\frac{x}{f(x)}\Big)^{k+1}=\operatorname{res}_0\Big(\frac{1}{f(x)}\Big)^{k+1}\\&=\frac{1}{2\pi i}\oint\Big(\frac{1}{f(x)}\Big)^{k+1}dx=\frac{1}{2\pi i}\oint\frac{1}{u^{k+1}}g'(u)du\\&=\operatorname{res}_0\Big(\frac{g'(u)}{u^{k+1}}\Big)=\text{coefficient of } u^k \text{ in } g'(u).\end{aligned}$$

(Integrating over a closed path around zero makes sense only for convergent power series with coefficients in $\mathbb{C}$, however the result holds for all power series with coefficients in $R \otimes \mathbb{Q}$.) Therefore,

$$g'(u) = \sum_{k \geq 0} \varphi[\mathbb{C}P^k] u^k.$$

This implies that g is the logarithm of the formal group law $\varphi(F_U)$, and thus f is its exponential. □

Taras Panov
Department of Geometry and Topology
Faculty of Mathematics and Mechanics
Moscow State University, Leninskie Gory
119991 Moscow RUSSIA

E-mail address: tpanov@higeom.math.msu.su