Lemma 6.5. *We have*

$$s_{i+j-1}[H_{ij}] = \begin{cases} j, & \text{if } i=0, \text{ i.e. } H_{ij} = \mathbb{C}P^{j-1}; \\ 2, & \text{if } i=j=1; \\ 0, & \text{if } i=1,\ j>1; \\ -\binom{i+j}{i}, & \text{if } i>1. \end{cases}$$

Proof. Let $i=0$. Since the stably complex structure on $H_{0j} = \mathbb{C}P^{j-1}$ is determined by the isomorphism $\mathcal{T}(\mathbb{C}P^{j-1}) \oplus \mathbb{C} \cong \bar{\eta} \oplus \ldots \oplus \bar{\eta}$ (j summands) and $x = c_1(\bar{\eta})$, we have

$$s_{j-1}[\mathbb{C}P^{j-1}] = jx^{j-1}\langle \mathbb{C}P^{j-1}\rangle = j.$$

Now let $i>0$. Then

$$s_{i+j-1}(\mathcal{T}(\mathbb{C}P^i \times \mathbb{C}P^j)) = (i+1)x^{i+j-1} + (j+1)y^{i+j-1} = \begin{cases} 2x^j + (j+1)y^j, & \text{if } i=1; \\ 0, & \text{if } i>1. \end{cases}$$

Denote by ν the normal bundle of the embedding $\iota\colon H_{ij} \to \mathbb{C}P^i \times \mathbb{C}P^j$. Then

$$\mathcal{T}(H_{ij}) \oplus \nu = \iota^*(\mathcal{T}(\mathbb{C}P^i \times \mathbb{C}P^j)). \tag{1}$$

Since $c_1(\nu) = \iota^*(x+y)$, we obtain $s_{i+j-1}(\nu) = \iota^*(x+y)^{i+j-1}$.

Assume $i=1$. Then by (1) and the previous Proposition,

$$\begin{aligned} s_j[H_{1j}] &= s_j(\mathcal{T}(H_{1j}))\langle H_{1j}\rangle = \iota^*(2x^j + (j+1)y^j - (x+y)^j)\langle H_{1j}\rangle \\ &= (2x^j + (j+1)y^j - (x+y)^j)(x+y)\langle \mathbb{C}P^1 \times \mathbb{C}P^j\rangle = \begin{cases} 2, & \text{if } j=1; \\ 0, & \text{if } j>1. \end{cases} \end{aligned}$$

Assume now that $i>1$. Then $s_{i+j-1}(\mathcal{T}(\mathbb{C}P^i \times \mathbb{C}P^j)) = 0$, and we obtain from (1) and the previous Proposition that

$$s_{i+j-1}[H_{ij}] = -s_{i+j-1}(\nu)\langle H_{ij}\rangle = -\iota^*(x+y)^{i+j-1}\langle H_{ij}\rangle = -(x+y)^{i+j}\langle \mathbb{C}P^i \times \mathbb{C}P^j\rangle = -\binom{i+j}{i}.$$

$\square$