Oberwolfach Seminar on Surgery Theory, 28 May - 1 June, 2012
Philipp Kühl's stenographic LATEX notes

Talk 1

1. Introduction

Let X be a finite (simplicial) CW-complex and M an n-dimensional closed Cat-manifold (Cat = Diff, PL, TOP).

Definition 1.1. $\mathscr{S}^{\mathrm{Cat}}(X) := \{f : M \simeq X\}/_{\simeq}$ where $f_0 : M_0 \simeq X \sim f_1 : M \simeq X$ if there exists an isomorphism $h : M_0 \simeq M_1$ such that

$$\begin{tikzcd} M_0 \arrow[r, "f_0"] \arrow[d, "h"] & X \\ M_1 \arrow[ur, "f_1"] & \end{tikzcd}$$

commutes.

Q1: When is $\mathscr{S}^{\mathrm{Cat}}(X) \neq \emptyset$?
Q2: How do we compute $\mathscr{S}^{\mathrm{Cat}}(X)$ if non-empty.
Re Q1: Look at 3 properties of manifolds.

(T) Existence of tangent bundles $TM \to M$, $TM \cong \mathbb{R}^n$

$$w : \pi = \pi_1(M) \to \mathbb{Z}/2 \text{ orientation character}$$

M orientable if and only if $w = 0$.

(PD) Existence of $[M] \in H_n(M; \mathbb{Z}^w)$ such that

$$\cap[M] : H^i(M; \mathbb{Z}[\pi]) \cong H_{n-i}(M; Z[\pi])$$

(Poincaré duality)

Definition 1.2. A Poincaré complex is a triple $(X, w, [X])$ where $w : \pi_1(X) \to \mathbb{Z}_2$, X is a finite CW complex and $[X] \in H_n(X; \mathbb{Z}^w)$ satisfies (PD)

Remark 1.3. In fact w is determined by X.

Observation 1. If $\mathscr{S}^{\mathrm{Cat}}(X) \neq \emptyset$ then X is a Poincaré complex.

Theorem 1.4. (L2) *Every Poincaré complex $X = (X, [X])$ admits a Spivak normal fibration (SNF) ν_X. This is a model for the normal bundle $f : M^n \to \mathbb{R}^{n+k}$.*

If $X \simeq M$ then ν_X has a bundle reduction ξ. Also we have the collapse map for f.

$$S^{n+k} \xrightarrow{c} \mathrm{Th}(\nu_M) = D(\nu_M)/S(\nu_M).$$

Moreover the SNF has a class $c : S^{n+k} \to \mathrm{Th}(\nu_X)$.

If ν_X has a reduction $\xi \simeq \nu_X$. Consider

$$c : S^{n+k} \to \mathrm{Th}(\nu_X) \to \mathrm{Th}(\xi)$$

After we make c transverse to X we get $c^{-1}(X) := M^n \subseteq S^{n+k}$ and hence a degree one normal map

$$\begin{tikzcd} \nu_M \arrow[r] \arrow[d] & \xi \arrow[d] \\ M_n \arrow[r, "f"] & X \end{tikzcd}$$

We define

$$\mathcal{N}(X) := \left\{ \begin{array}{ccc} \nu_M & \longrightarrow & \xi \\ \downarrow & & \downarrow \\ M_n & \xrightarrow{f} & X \end{array} \right\}_{/\simeq \text{ normal bordism}}$$

Surgery gives a map

$$\sigma\colon \mathcal{N}(X) \to L_n(\mathbb{Z}\pi, w)$$

such that

$$\sigma = 0 \Leftarrow (M, f, \overline{f}) \sim (M', f', \overline{f'}) \text{ with } f' \text{ a htpy eq}$$
$$\Rightarrow \qquad (n \geq 5)$$

Thus $\mathscr{S}^{\mathrm{Cat}}(X) \neq \emptyset$ if and only if there exists (ξ, c) such that

$$\sigma(\overline{f}_{\xi,c}, f_{\xi,c}) = 0 \in L_n(\mathbb{Z}\pi).$$

If $\mathscr{S}^{\mathrm{Cat}}(X) \neq \emptyset$ we consider the uniqueness question.

Theorem 1.5 (B-N-SW ...)**.** *There exists a long exact sequence of pointed sets*

$$\longrightarrow L_{n+1}(\mathbb{Z}\pi) \longrightarrow \mathscr{S} \longrightarrow \mathcal{N}(X) \xrightarrow{\sigma} L_n(\mathbb{Z}\pi)$$

Theorem 1.6 (R)**.** *This sequence has an algebraic twin when Cat = TOP*

$$\begin{array}{ccccccccc} \longrightarrow & L_{n+1}(\mathbb{Z}\pi) & \longrightarrow & \mathscr{S}^{TOP} & \longrightarrow & \mathcal{N}^{\mathrm{TOP}}(X) & \xrightarrow{\sigma} & L_n(\mathbb{Z}\pi) \\ & \downarrow = & & \downarrow \cong & & \downarrow \cong & & \downarrow = \\ \longrightarrow & L_{n+1}(\mathbb{Z}\pi) & \longrightarrow & \mathbb{S}_{n+1}(X) & \longrightarrow & H_n(X; \mathbb{L}_\bullet\langle 1\rangle) & \longrightarrow & L_n(\mathbb{Z}\pi) \end{array}$$

2. Bundles

B = base-space, finite CW-complex.
Recall: $\mathrm{Vect}^k(B) := \{[E] \mid E \to B \text{ a rank k vector bundle}\}$.
Assume E has a metric.

Theorem 2.1 (Husemoeller)**.** *There exists a universal vector bundle*

$$VO(k) \to \mathrm{BO}(k)$$

and a bijection

$$Vect_k(B) \cong [B, \mathrm{BO}(k)]$$
$$[f^* \, VO(k)] \leftarrow\!\shortmid [f]$$

Example 2.2.
(1) $B \times \mathbb{R}^k =: \underline{\mathbb{R}^k}$
(2) TM, M a smooth manifold $M^n \looparrowright V^{n+k}$ f an immersion
(3) $\Rightarrow f^*TV \cong TM \oplus \nu_f$ (ν_f the universal bundle of f)
(4) $f\colon M^n \hookrightarrow \mathbb{R}^{n+k}$, $\quad k >> n$. Then ν_f is the stable normal bundle of M. The isomorphism class of ν_f is independent of $f \Rightarrow \nu_M$. $\nu_M \oplus TM \cong \mathbb{R}^{n+k}$

Definition 2.3. A stable vector bundle over X is a sequence $\{E_i, \alpha_i\} = \xi$ where $\alpha_i E_{i+1} \cong E_i \oplus \underline{\mathbb{R}}$

A stable vector bundle gives rise to a Thom-Spectrum.

$$M(\xi) = \{\mathrm{Th}(E_i), \text{obvious maps}\}.$$

Recall

- $\mathrm{Th}(E_i) \cong D(E_i)/S(E_i)$
- $\mathrm{Th}(E \oplus \underline{R}) \cong \Sigma\, \mathrm{Th}(E)$

Example 2.4. $\mathrm{MO} \cong M(\xi)$ where $\xi = \{\mathrm{VO}(k)\}$

Here we have $\mathrm{VO}(k) \oplus \underline{\mathbb{R}} \to \mathrm{VO}(k+1)$

Theorem 2.5 (Pontrjagin-Thom)**.** *There exists an isomorphism*

$$\pi_n^s(M(\xi)) \cong \Omega_n(\xi)$$

where π_n^s are the stable homotopy groups of the spectrum and $\Omega_n(\xi)$ bordism of manifolds with ξ-structure. It consists of equivalence classes of the following

$$\begin{tikzcd}
 & \nu_M \arrow[r, "\overline{f}_k"] \arrow[d] & \xi_k \arrow[d] \\
S^{n+k} & M \arrow[l, hook'] \arrow[r, "f_k"] & X_k
\end{tikzcd}$$

Idea of proof: Make c transverse to X_k. So $M := c^{-1}(X_k)$ has $\nu_m \cong c|_M^* E(\xi_k)$. Apply transversality to a map

$$S^{n+k} \xrightarrow{c} \mathrm{Th}(E(\xi_k))$$

3. Microbundles

Let E, B be spaces.

Definition 3.1. A microbundle ξ of rank k is a triple (E, B, i, j) where $B \xrightarrow{i} E \xrightarrow{j} B$ have

1. $j \circ i = \mathrm{id}_B$
2. for all $b \in B$ exist neighbourhoods $U \ni b$ and $V \ni i(b)$ with $i(U) \subseteq V$ and $j(V) \subseteq U$ and there exists a homeomorphism $h \colon V \cong U \times \mathbb{R}^k$ such that the following diagram commutes

$$\begin{tikzcd}
 & V \arrow[dr, "j|V"] \arrow[dd] & \\
U \arrow[ur, "i|U"] \arrow[dr, "\times 0"'] & & U \\
 & U \times \mathbb{R}^n \arrow[ur, "p_i"'] &
\end{tikzcd}$$

Define $(E_1, B, i_1, j_1) \cong (E_2, B, i_2, j_2)$ when there exist neighbourhoods $V_a \supseteq i_a(B)$ (a=0,1) and a homeomorphism $h \colon V_1 \cong V_2$ such that the following diagram commutes

$$\begin{tikzcd}
 & V_1 \arrow[dr, "j|V_1"] \arrow[dd] & \\
B \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & B \\
 & V_2 \arrow[ur, "j_2|V_2"'] &
\end{tikzcd} .$$

Example 3.2. M a topological manifold and form $\xi_M = (M \times M, M, \Delta_M, p)$

$$x \longmapsto (x, x)$$
$$M \xrightarrow{\Delta_M} M \times M \xrightarrow{p_1} M$$
$$(x, y) \longmapsto x$$

This is a micro bundle.

Example 3.3. If $\underline{\mathbb{R}}^k \to E \xrightarrow{\pi} B$ is a linear vector bundle then (E, B, s, π) is a micro bundle.

Theorem 3.4 (Milnor). *If M is a smooth manifold then $TM \cong \xi_M$ is a micro bundle.*

Talk 2

4. Topological bundles

Recall. The frame bundle of a vector bundle E^k

$$\begin{aligned} \mathrm{FE} &\subseteq E \times \ldots \times E \\ &= \{(v_1, \ldots v_k) \mid \text{ such that this is an o.n basis of some } E_x\} \end{aligned}$$

Observe that there exists a continues action of $O(k)$ on FE and a principal $O(k)$-bundle

$$O(k) \to \mathrm{FE} \to X$$

and $E \cong \mathrm{FE} \times_{O(k)} \mathbb{R}^k$ (Borel product)

For any topological group G, there exists a universal bundle $G \to \mathrm{EG} \to \mathrm{BG}$ with $\mathrm{EG} \simeq *$. If G acts on a space F we obtain the universal bundle

$$F \to \mathrm{EG} \times_G F \to BG$$

of F-bundles with structure group G.

Example 4.1. For $G = \mathrm{Homeo}_*(\mathbb{R}^k) =: \mathrm{TOP}_n$ we have topological $\mathbb{R}^n$-bundles

$$\mathrm{Homeo}_*(\mathbb{R}^k) = \left\{f\colon \mathbb{R}^k \cong \mathbb{R}^k \mid f(\underline{0}) = \underline{0}\right\}$$

It follows that there exists spaces $\mathrm{ETOP}_k \to \mathrm{BTOP}_k$ and $\mathbb{R}^k \to \mathrm{VTOP}_k \to \mathrm{BTOP}_k$.

Observe

$$\begin{aligned} \mathrm{TOP}_k &\to \mathrm{TOP}_{k+1} \\ f &\mapsto f \times \mathrm{id}_{\underline{R}} \end{aligned}$$

We define $\mathrm{TOP} := \lim_{k\to\infty} TOP_k$ the notion of stable bundles and $\mathrm{BTOP} := \lim_{k\to\infty} BTOP_k$

Theorem 4.2 (Kister, (Lashof and Rothenberg), Mazur?). *Let $B \xrightarrow{i} E \xrightarrow{j} B$ be a microbundle. Then there exists $E_i \subseteq E, E_i \supset i(B)$ such that*

1. *$E_i = E_i(\xi)$ is the total space of a topological $\mathbb{R}^k$-bundle over B.*
2. *the inclusion $E_i \to E$ is an isomorphism of microbundles*
3. *E_i is uniquely determined up to $\mathbb{R}^k$-bundle isomorphism.*

"Microbundles are bundles"

Consequence: Every TOP manifold M^n has

$$M \times M \supseteq TM \to M \to BTOP_n$$

an ??? $\mathbb{R}^n$-bundle, it's tangent bundle.

Theorem 4.3 (Kirby-Siebenmann ($n \neq 4$), Freedman-Quinn ($n = 4$)). *Transversality holds for TOP_n-bundles, i.e. given $c\colon M^{n+k} \to E(\xi)$, M^{n+k} a TOP-manifold, ξ is an $\mathbb{R}^k$-bundle such that $c|_{\partial M} \pitchfork X$ i.e. $(c|_{\partial M})^{-1}(X) \subseteq \partial M$ is a manifold of* *codimension k, locally flatly embedded. Then $c \simeq_{rel \partial M} c'$ such that $c' \pitchfork X$. Locally flat means that for all $x \in N$ there exits $U \ni x$ and $V \ni F(x)$ and homeos*

$$\begin{tikzcd} U \arrow[r, "\cong"] \arrow[d, "c'"'] & \mathbb{R}^n \arrow[d, "\times 0"'] \\ V \arrow[r, "\cong"] & \mathbb{R}^{n+k} \end{tikzcd}$$

Moreover $N := (c')^{-1}(X)$ has an $\mathbb{R}^k$-normal bundle isomorphic to $(c')^*\xi$

Example 4.4. Sanderson 66. There exist topological embeddings without normal bundle.

Corollary 4.5. *There exists a Pontrjagin-Thom isomorphism in TOP*

$$\Omega_n(\xi) \cong \pi_n(\mathrm{Th}(\xi))$$

ξ a stable topological bundle.

Remark 4.6. If ξ has a vector bundle reduction then ξ'

$$\Omega_n^{\mathrm{TOP}}(\xi) \cong \pi_n(\mathrm{Th}(\xi)) \cong \pi_n(\mathrm{Th}(\xi')) \cong \Omega_n^{\mathrm{Diff}}(\xi')$$

where if $\mathbb{R}^k \to E \to X$ is a topological $\mathbb{R}^k$-bundle then $\mathrm{Th}(E) = E^+$ if X compact.

5. Spherical fibrations

A spherical fibration of rank k ξ is a fibration

$$S^{k-1} \to S(\xi) \to X$$

with a homotopy fibre S^{k-1}. Note that $D(\pi) := \mathrm{cyl}(\pi)$ is a "disc bundle". is a fibration with homotopy fibre D^k

Example 5.1. If ξ is a vector bundle, form $S(\xi)$ and forget to regard it as a fibration. If ξ is an $\mathbb{R}^k$-bundle, set $S(\xi) = E(\xi) - s_0(X)$ where s_0 is a zero-section.

Definition 5.2.
- fibre homotopy equivalence:

$$\xi_0 \simeq \xi_1 \iff \exists\, f : \begin{tikzcd} S(\xi_0) \arrow[r, "\simeq"] \arrow[d] & S(\xi_1) \arrow[d] \\ X \arrow[r, "="] & X \end{tikzcd}$$

- $SF_k(X) \cong \{[E(\xi)] \mid \xi$ a rank k spherical fibration $\}$

Example 5.3. $X \times S^{k-1} \to X$ the trivial spherical fibration. Fibre-wise join with $x \times S^0$ defines stabilisation. $\Rightarrow$ stable spherical fibration $\{\xi_i, \alpha_i\}$. $\alpha_i : E(\xi_{i}, *_X S^{0)} \to E(\xi_{i+1})$.

$$\mathrm{Th}(\xi) := D(\xi)/S(\xi)$$

Can use this to define $\mathrm{Th}(\xi)$ if ξ an $\mathbb{R}^k$-bundle. Hence we have forgetful maps

$$\mathrm{VB}_k(X) \to VTOP_k(X) \to F_k(X)$$

Stablise

$$SVB(X) \to \mathrm{SVTOP}(X) \to SSF(X)$$

Remark 5.4. Spherical fibrations are not bundles in general and hence don't have a structure group. But they do have a structure monoid:

$$G_k := \mathrm{Map}_{\pm}(S^{k-1}, S^{k-1})$$
$$G := \lim_{k \to \infty} G_k.$$

There exist forgetful maps

$$\begin{array}{ccccc} O(k) & \longrightarrow & \mathrm{TOP}(k) & \longrightarrow & G_{k+1} \\ \downarrow & & \downarrow & & \downarrow \\ O & \longrightarrow & TOP & \longrightarrow & G \end{array}$$

Given $E(\xi)$ a spherical fibration rank k define

$$\begin{aligned} P(\xi) &\subseteq \mathrm{Map}(S^{k-1}, E(\xi)) \\ &:= \left\{ f \colon \xrightarrow{\simeq} E(\xi)_x \right\} \end{aligned}$$

Observe that G_k acts by precomposition on $P(\xi)$. We get $G_k \to P(\xi) \to X$ ("monoid fibration")

Theorem 5.5 (Stasheff)**.** *There exists a spherical fibration* $\mathrm{VG}_k \to \mathrm{BG}_k$ *and a bijection*

$$\begin{aligned} \mathrm{SF}_k(X) &\cong [X, BK_k] \\ [f^* \mathrm{VG}_k] &\leftarrow\!\shortmid [f] \end{aligned}$$

Moreover $\mathrm{SSF}(X) \cong [X; \mathrm{BG}]$

Lemma 5.6 (Madsen-Milgram)**.** *There exists a fibration sequence*

$$\begin{aligned} F_k \to G_{k+1} &\xrightarrow{ev} S^k \\ f &\mapsto f(x_0) \end{aligned}$$

where $F_k \simeq \Omega^k_{+1} S_k \cup \Omega^k_{-1} S^k$ *(use adjunction)*

Corollary 5.7.

$$\pi_i(G_k) \cong \pi_i(F_k) \cong \pi_{k+1}(S^k) \cong \pi_i^s \quad i \le k-2$$

Definition 5.8. The induced map

$$J_i \colon \pi_i(O) \to \pi_i(G) \cong \pi_i^s$$

is the (stable) J-homomorphism.

Theorem 5.9 (Bott)**.**

	0	1	2	3	4	5	6	7
$\pi_i(O) \cong$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	0	0	$\mathbb{Z}$

Theorem 5.10 (Adams)**.**

$$J_i \colon \pi_i(O) \to \pi_i(G)$$

injective for (*i=0,1.*) $J_{4k-1}(\mathbb{Z})$ *is a cyclic summand of Bernoulli order denom* $(\frac{B_k}{4k})$

Theorem 5.11 (Serre)**.** $\pi_i(G) < \infty$ *if* $i > 0$

Theorem 5.12 (K-M)**.** *There exists a short exact sequence for* $i \ge 5$

$$\pi_i(O) \to \pi_i \mathrm{PL} \to \pi_i PL/O$$

where $\pi_i \mathrm{PL}/O$ *is finite.*

Theorem 5.13 (Arf)**.** $\pi_i(\mathrm{PL}/O)$ *is 6-connected.*

Theorem 5.14 (Kirby-Siebenmann). $\mathrm{TOP}/\mathrm{PL} \simeq K(\mathbb{Z}_2, 3)$ *Eilenberg-MacLane space.* $\Rightarrow$

$$\ldots \to \pi_i(\mathrm{TOP}) \to \pi_i(G) \to \pi_i(\mathrm{G/TOP}) \to \ldots$$

$$\otimes\mathbb{Q} \Rightarrow \pi_{i+1}(\mathrm{G/TOP}) \otimes \mathbb{Q} \cong \pi_i(\mathrm{TOP}) \otimes \mathbb{Q} \cong \pi_i(O) \otimes \mathbb{Q}$$

In fact, surgery will show us that $(i \geq 5)$

$$\pi_i(\mathrm{G/TOP}) \cong L_i(\mathbb{Z}) \cong \begin{cases} 0 & i \text{ odd} \\ \mathbb{Z}_2 & i = 4k+2 \\ \mathbb{Z} & i = 4k \end{cases}$$

Theorem 5.15 (Boardman-Vogt). *There exists* Ω*-loop structures on*

$$O \to \mathrm{PL} \to \mathrm{TOP} \to G$$

and ∞*-loop maps. Hence* $G/\mathrm{CAT}, \mathrm{TOP}/\mathrm{CAT}$ *are compatible* ∞*-loop spaces*

Two explicit descriptions of $J_i \colon \pi(O) \to \pi_i(G) \cong \pi_i^S$ (induced map of spaces)
$\alpha \in \pi_i(O_{k+1}) \rightsquigarrow f_\alpha \colon S^i \to \mathrm{Map}_\pm(S^k, S^k)$ "Adjoin":

$$F_\alpha \colon S^i \times S^i \longrightarrow S^k,$$
$$(x,y) \mapsto f_\alpha(x)(y)$$

$J(\alpha) = [\Sigma F_\alpha]$ in the stable range $\Rightarrow$

$$\Sigma F_\alpha \colon S^i * S^k \longrightarrow \Sigma^k,$$
$$S^{i+k+1} \mapsto S^k + 1$$

$$\pi_i(O) \to \pi_i(G) \cong \pi_i^S \cong \Omega_i^{\mathrm{fr}}$$

For $k \geq i+2$ $\alpha \in \pi_i(O_k)$, take $S^i \times D^k \hookrightarrow S^{i+k}$ with standard framing (ie. extends over $D^{i+1} \times D^k \hookrightarrow D^i k + 1$.)

$$[(S^i, F^\alpha)] = J(\alpha)$$

Exotic framing. Now twist by α_i

Lemma 5.16 (Milnor). *Let* $\xi\alpha \to S^i$ *be a rank* k *vector bundle. then* $\mathrm{Th}(\xi\alpha) \simeq S^k \cup_\emptyset e^{i+k}$ $\alpha \in \pi_{i-1}(\mathrm{SO}_k) \xrightarrow{J} \pi_{i+k-1}(S^k)$ *where* $\phi \colon S^{i+k-1} \to S^k$*.* $\phi = J(\alpha)$

Talk 3

6. The Spivak Normal Fibration

Definition 6.1. $(X, [X])$ a Poincaré complex of formal dimension n. A Spivak normal fibration (SNF) for X is a stable spherical fibration

$$\xi \to X$$

which admits a Spivak Normal Structure, ie. a map

$$c \colon S^{n+k} \to \mathrm{Th}(\xi_k)$$

such that $c_*[S^{n+k}]$ generates $H_{n+k}(\mathrm{Th}(\xi_k))$

We are interested in such pairs (ξ, c).

Remark 6.2. Thom isomorphism

$$H_{n+k}(\mathrm{Th}(\xi_k)) \cong H_n(X; \mathbb{Z}^w)$$
$$c_*[S^{n+k}] \mapsto \pm[X]$$

Theorem 6.3 (Browder-Spivak)**.** *SNS's (hence SNF's) always exist and are unique in the following sense: Given* (ξ_0, c_0) *and* (ξ_1, c_1) *up to fibre homotopy* $\exists ! af.h.e.$ $f : \xi_0 \cong \xi_1$ *such that*

$$[\mathrm{Th}(f) \circ c_0] = [c_1] \in \pi_{n+k}(\mathrm{Th}(\xi_1))$$

Lueck. Existence when $\pi = e$.

- Realise X as a simplicial complex. $\Rightarrow X \hookrightarrow R^{n+k}, \quad k >> n$
- there exists a regular neighbourhood of $X \hookrightarrow \mathbb{R}^{n+k}$ i.e. $X \subseteq N \subseteq \mathbb{R}^{n+k}$ where N is a codimension 0 submanifold of $\mathbb{R}^{n+k}$. and

$$\begin{tikzcd} X \arrow[r, hook, shift left, "i"] & N \arrow[l, shift left, "q"] \\ & \partial N \arrow[ul, "p"] \arrow[u, hook, "\subseteq"'] \end{tikzcd}$$

q a homotopy inverse to i. $X \to N$ is a homotopy equivalence. Thus $\partial N \xrightarrow{p} X$ is our candidate for the SNF: after we convert it to a fibration.

$$\begin{tikzcd} N \arrow[d] & \partial N \arrow[l, hook'] \arrow[r] \arrow[d] & X \\ \mathrm{cyl}(p) & E_p \arrow[l, hook', "\supseteq"'] \arrow[r] & X \end{tikzcd}$$

One checks that $(N, \partial N) \to (\mathrm{cyl}(p), E_p)$ is a homotopy equivalence of pairs

$$\begin{tikzcd} H^{n-+}(X) \arrow[r, "\cap[X]"] \arrow[d] & H_p(X) \arrow[d, "i_*"] \\ H_{n+k+p}(N, \partial N) \arrow[r, "\cap[N]"] & H_p(N) \end{tikzcd}$$

Choose $u \in H^k(N, \partial N)$ to be the Poincaré dual of $[X]$ after identifying

$$i_* \colon H_n(X) \longrightarrow H_n(N),$$
$$[X] \simeq i_*[X]$$

$\square$

Exercise 6.4. If $(D, S) \to X$ is a fibration over a 1-connected base space whit a Thom class then $S \to X$ is a spherical fibration. Hint: Use the Leray-Serre spectral sequence of a pair.

7. Spanier-Whitehead Duality

Recall if $A \subseteq \mathbb{R}^n$ is a compact subspace and $C = \mathbb{R}^n \setminus A$ is the complement then $\widetilde{H}_{i(C)} \cong H^{n-i-1}(A)$. Apply Poincaré duality to a regular neighbourhood of A.

Definition 7.1. Let $A \subseteq S^n$ be a finite compact CW complex. Then the S-dual of A is a finite CW-complex onto which $C_A = S^n \setminus A$ retracts.

Example 7.2.

$$S^p \hookrightarrow S^{p+q-1} \hookleftarrow S^q$$

C_A S-dual of S^p

This leads to a contra-variant functor

$$D \colon \mathscr{FS} \to \mathscr{FA} \text{ finite spectra}$$
$$(\mathscr{FC} \to \mathscr{FC} \text{ finite CW complexes})$$

Theorem 7.3 (Milnor, Spanier for manifolds)**.** *The S-dual of X_+ is* $\mathrm{Th}(\nu_X)$ *for X a Poincaré complex.*

Proof. Assume that $k >> n$ so thath in fact we had $N' \hookrightarrow S^{n+k-1}$ and $N = N' \times I$. By general position $\pi_1(C_X) = 0$. We know its homology groups by Alexander duality.

$$\partial N = N' \cup N' \to C_X$$

$H_*(N') = n <$ connectivity of C_X It follows $j|_{N'} N'$ is null homotopic. Hence there exists a map

$$\partial N \cup \mathscr{C}(N') \to C_X - \mathrm{pt}$$

This map is a H_*-isomorphism of simply connected spaces. Hence a homotopy equivalence. But $(\partial N, N') \simeq (\nu_X, s_0(X))$. $s_0(X)$ is a section since $\nu_X^k = (\nu_X^{k-1} * \underline{S^0})$ Thus $\mathrm{Th}(\nu_X^{k-1}) \simeq \nu_X/\mathscr{C}(s_0(X)) \simeq C_X - \mathrm{pt}$ □

8. A more sophisticated Alexander Duality

Say X and Y are N-dual if there exists a map

$$p\colon S^N \to X \wedge Y$$

such that $\backslash p_*[S^N]\colon H^p(X) \to H_{N-p}(Y)$ is an isomorphism.

We can realise this map for X_+ and $\mathrm{Th}(\nu_X^k)$ as follows:

$$S^{n+k} \xrightarrow{c} \mathrm{Th}(\nu_X) \xrightarrow{\Delta'} \mathrm{Th}(\nu_X) \wedge D(\nu_X)_+ \xrightarrow{1\wedge p} \mathrm{Th}(\nu_X) \wedge X_+$$

where $(D(\nu_X), S(\nu_X)) \xrightarrow{\Delta} (D(\nu_X) \times D(\nu_X), S(\nu_X) \times D(\nu_X))$. This induces Δ'.

Exercise 8.1. p as above presents X_+ and ν_X as $(n+k)$-duals.

9. Normal bordism

Setting: X is a Poincaré complex of dimension n and ν_X has a bundle reduction ξ i.e. there exists a fibre homotopy equivalence $h\colon \nu_X \simeq \xi$.

Definition 9.1 (Lueck - 4 notions of a normal invariant)**.** (1)

$$\mathcal{N}(X) \ni \left\{ \begin{array}{c|c} \begin{array}{ccc} \nu_M & \xrightarrow{\bar f} & \xi \\ \downarrow & & \downarrow \\ M^n & \xrightarrow{f} & X \end{array} & (M, \overline{f}, f) \text{ a degree one normal map} \end{array} \right\}$$

$\simeq$ generated by normal bordism of (X, ξ) IMAGE3

(ii) If

$$\begin{array}{ccc} \alpha\colon \xi & \xrightarrow{\cong} & \xi \\ \downarrow & & \downarrow \\ X & \xrightarrow{\cong} & X \end{array}$$

is a Cat-bundle isomorphism then $(M, \overline{f}, f) \sim (M, \alpha \circ \overline{f}, f)$. Note: $\mathcal{N}(X) = \amalg_{[\xi]} \mathcal{N}^{\mathrm{fr}}(X, \xi)$. ξ runs over all stable cat fibre homotopy equivalent to ν_X. $\mathcal{N}^{\mathrm{fr}}(X, \xi)$ is defined as above for fixed ξ wihtout (2)

(2) As for (1) but replace ν_M by τ_M, the stable tangent bundle of M

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}^a} & \xrightarrow{\bar f} & \xi \\ \downarrow & & \downarrow \\ M^n & \xrightarrow{f} & X \end{array}$$

(3)

$$\mathscr{I}(X) = \left\{ (\xi, c) \;\middle|\; \begin{array}{c} \xi \text{ is a stable Cat bundle reducing} \\ \nu_X \text{ and } c \in \pi_{n+k}(\mathrm{Th}(\xi^k)) \text{ satisfies} \\ c_*[S^{n+k}] \text{ generates } H_{n+k}(\mathrm{Th}(\xi^k)) \end{array} \right\}_{/\sim}$$

where $(\xi_0, c_0) \sim (\xi_1, c_1) \Leftrightarrow \exists$ Cat bundle isomorphism $\alpha\colon \xi_0 \cong \xi_1$ such that $\mathrm{Th}(\alpha)_*[c_0] = [c_1]$

(4) $[X, G/\mathrm{Cat}]$ homotopy classes of maps X to G/Cat where G/Cat is the homotopy fibre of $\mathrm{BCat} \to BG$.

Talk 4

10. Normal bordism (cont.)

Setting: X is a Poincaré complex of dimension n and ν_X has a bundle reduction ξ i.e. there exists a fibre homotopy equivalence $h\colon \nu_X \simeq \xi$.

Recall $\mathcal{N}(X), \mathcal{N}^T(X), \mathscr{I}(X), [X, G/\mathrm{Cat}]$.

Theorem 10.1 (Browder-Novikov, Sullivan)**.** *There are isomorphisms*

(1) $\mathcal{N}(X) \cong \mathscr{I}(X)$
(2) $\mathcal{N}(X) \cong \mathcal{N}^T(X)$
(3) $\mathcal{N}(X) \cong [X, G/Cat]$

Proof. (1) Apply transversality.

$$[c\colon S^{n+k} \to \mathrm{Th}(\xi_X)] \mapsto \left[\begin{array}{ccc} \nu_M & \xrightarrow{\bar f} & \xi \\ \downarrow & & \downarrow \\ M^n & \xrightarrow{f} & X \end{array} \right]$$

where $M \coloneqq c^{-1}(X)$. Make $c \pitchfork X \subseteq \mathrm{Th}(\xi_k)$. NB there we need to apply Cat-transversality. Well-defined. Apply transversality to a homotopy IMAGE4 $C^{-1}(X)$ gives a normal bordism. Bundle automorphism condition checks out. The inverse is given by collapsing

(2) Special case $X = M$ is a Cat-manifold. What are maps to G/Cat? IMAGE 5 Now take $g \coloneqq h^{-1}$ and make g transverse to $s_0(X) \subseteq X$. Define

$$\left[\begin{array}{ccc} \nu_{M_f} \cong g^*E \oplus \nu_{X \times \mathbb{R}^k} & \xrightarrow{\bar j} & E \oplus \nu_Y \\ \downarrow & & \\ M_f \coloneqq g^{-1}(X) & \xrightarrow{j} & X \end{array} \right] \leftarrow\!\shortmid [f]$$

But $\nu_E \cong \nu_X$

(3) $\mathcal{N}(X) \to \mathcal{N}^T(X)$. Since X is compact, there exists an inverse η such that $\eta \oplus \xi \cong \underline{\mathbb{R}}^q$.

$$\begin{array}{ccc} TM \oplus \nu_M & \longrightarrow & \eta \oplus \xi \cong \underline{\mathbb{R}^k} \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

Take $f^*\eta \oplus \underline{\mathbb{R}}^{n+k} \cong f^*\eta \oplus TM \oplus \nu_M \cong f^*\eta \oplus TM \oplus f^*\xi \cong TM \oplus \mathbb{R}^q$

Correction: We see that the bundles in

$$\begin{tikzcd} \mathcal{N}^T(X) = \tau_M \arrow[r] \arrow[d] & \eta \arrow[d] \\ M \arrow[r] & X \end{tikzcd}$$

η reduces ν_X^{-1}. in fact, from the uniqueness of the SNF this is automatic □

Proposition 10.2. *If* $\begin{tikzcd} \nu_M \arrow[r, "\bar{f}"] \arrow[d] & \xi \arrow[d] \\ M^n \arrow[r, "f"] & X \end{tikzcd}$ *is degree one then* ξ *reduces* ν_X

Goal: To define $\sigma\colon \mathcal{N}(X) \to L_n(\mathbb{Z}\pi)$

Remark 10.3. If $X = M$ then $[\overline{\mathrm{id}}, \mathrm{id}, M] \in \mathcal{N}(X)$ gives a base-point. In general $[X, G/\mathrm{Cat}]$ acts freely and transitively on $\mathcal{N}, X$. If X does not have a manifold structure or if one is not fixed it only acts.

11. Surgery below the middle dimension

Lemma 11.1. *Any degree one normal map* $(M, \overline{f}, f)$ *is normally bordant to* $(M', \overline{f}', f')$ *where* $f'\colon M \to X$ *is a* q*-equivalence.* $(n = 2q$ *or* $2 + 1)$

Proof. Proceed by induction on q.

π_0 : Assume X compact. We require 0-surgeries IMAG 6

π_1 (Kreck): $\pi_1(X) = \langle x_1, \dots, x_k \mid r_1, \dots r_s\rangle \xrightarrow{w} \mathbb{Z}\pi$. For each x_i take the manifold $S^{n+1} \tilde{\times} =: M_i \to S' \to X$. Degree one normal maps

$$\begin{tikzcd} M_1 \amalg \dots \amalg M_k \arrow[dr, "f_k"] & \\ M \arrow[r, "f"] & X \end{tikzcd}$$

Apply 0-surgery $M' = M \# M_1 \dots \# M_k \xrightarrow{f'} X$ Assume $n \geq 3$. It follows $\pi_1(M) \cong \pi_1(M) * *_i = 1^k \mathbb{Z}(z_i)$. $\pi_1(M') = \langle a_1, \dots, a_j, z_1, \dots, z_k \mid R_1, \dots R_p\rangle$. Write $f_*(a_i) = w(x_{1(}\dots, x_k)) \in \pi_1(X)$. $z_i \mapsto x_i$. then

$$b_i \begin{cases} a_i^{-1} w(z_1, \dots, z_k) \mapsto 0 \in \pi_1(X) \\ \text{and } r_i(z_1, \dots, z_k) \mapsto 0 \in \pi_1(X) \end{cases}$$

Now the $b_i \in \ker(f')\colon \pi_1(M') \to \pi_1(X)$. It follows $w(b_i) = 0$ (degree one normal map) and hence b_i is represented by $D^{n-1} \times S^1 \hookrightarrow M'$ embedding in general position.

$$\pi_1(\mathrm{TOP}) \simeq \pi_1(\mathrm{TOP}) \cong \pi_1(O) \cong \mathbb{Z}_2 \quad (k \geq 2)$$

Construct the following normal bordism IMAGE 7 We need to take care that we can extend the normal data to W. Note that $M'' \to\xrightarrow{f''} X$ is a π_1-isomorphism. Why? $W \simeq M' \cup (\cup D^2), \simeq M'' \cup (\cup D^{n-1})$.

$$\begin{tikzcd} \pi_1(W) \cong \pi_1(M')/\langle b_i\rangle \cong pi(M'') \arrow[d, two heads] \\ \pi_1(X) \end{tikzcd}$$

$(n-1 \geq 3)$ Also an injection. Now work inductively $f\colon M \to X$. $p_i(f) = 0$ for $i \leq k < q$. Suffices to kill an element of $\pi_{k+1}(f) \ni \alpha$. Represent α by a diagram.

$$\begin{tikzcd} S^k \arrow[r] \arrow[d] & M \arrow[d] \\ D^{k+1} \arrow[r] & X \end{tikzcd}$$

Cover with bundle maps

$$\begin{tikzcd} T(S^k \times D^{n-k}) \oplus \underline{\mathbb{R}}^{a+b} \arrow[r] \arrow[d] & TM \oplus \underline{\mathbb{R}}^{a+b} \arrow[d] \\ T(D^{k+1} \times D^{n-k}) \oplus \underline{\mathbb{R}}^{a+b-1} \arrow[r] & \xi \oplus \underline{\mathbb{R}}^{b} \end{tikzcd}$$

$\square$

Talk 5

12. Surgery below the middle dimension (cont.)

Recall.

$$\begin{tikzcd} \nu_M \arrow[r, "\bar{f}"] \arrow[d] & \xi \arrow[d] \\ M^n \arrow[r, "f"] & X \end{tikzcd}$$

$(M, \overline{f}, f)$ is normally bordant to $(M', \overline{f}, f)$ with f' a q-equivalence.

Idea: to work inductively to kill $\pi_{k+1}(f)$

$$\begin{tikzcd} D^{n-k} \times S^k \arrow[r, "\alpha"] \arrow[d] & M \arrow[d] \\ D^{n-k} \times D^k + 1 \arrow[r, "\alpha'"] & X \end{tikzcd} \qquad \begin{tikzcd} T(D^{n-k}) \oplus \underline{\mathbb{R}}^q \arrow[r, "\alpha"] \arrow[d] & T \\ T(D^{n-k} \times D^{k+1)} \oplus \underline{\mathbb{R}}^{q-1} \arrow[r, "\alpha'"] & {} \end{tikzcd}$$

$$\alpha'^{*}\xi = \underline{\mathbb{R}}^j \qquad T(D^{n-k} \times D^{k+1)} = \underline{\mathbb{R}}^n$$

Theorem 12.1 (Hirsch-Smale)**.** *Let M^n and N^n be smooth manifolds, N closed. Assume M has a handle decomposition with q-handles $(q \leq n-2)$. If $1 \leq m \leq n$ then taking the differential gives a bijection*

$$\underbrace{\pi_0 \operatorname{Imm}(M, N)}_{\text{regular htpy classes of immersions}} \cong \pi_0 \operatorname{Mono}(TM \oplus \underline{\mathbb{R}}^q, TN \oplus \underline{\mathbb{R}}^q)$$

$$[f\colon M \looparrowright N][Tf \oplus \mathrm{id}_{\underline{\mathbb{R}}^q}]$$

Remark 12.2. (1) This is a version of the h-principal
(2) At least if $m = n$ a similar statement holds in TOP

How to proceed to complete the surgery step:

For $k < q$, we can assume that $\alpha\colon S^k \to M$ is an embedding and the bundle data ensure that α has a trivial bundle.

IMAGET5-1

All bundle data are compatible.

$$W := \text{trace of the surgery on } \alpha + \text{bundle data}$$

$W \simeq M \cup e^k$ and $W \simeq M' \cup e^{n-k} \Rightarrow \pi_k(M') \cong \pi_k(W) \cong \pi_k(M)/\langle\alpha\rangle$ ($\mathbb{Z}\pi$-module generated by α)

Exercise 12.3. $\pi_{k+1}(f) \cong H_{k+1}(\widetilde{X}, \widetilde{M}; \mathbb{Z})$ always f.g. over $\mathbb{Z}\pi$

13. The middle dimension $n = 2q$

Definition 13.1.

$I_k(M) := \{(f,w)\} =$ reg. htpy classes of immersion $f\colon \looparrowright M$ and paths $w\colon b \rightsquigarrow f(s)$ and htpy of paths

IMAGE-2

$$I_k^0(M) \subseteq I_k(M), \text{immersion with trivial normal bundle}$$

Exercise 13.2.
- $I_q^0(M) \xrightarrow{\cong} \pi_q(M)$
- But using the path w, we have a geometric definition of $+$ in $I_q(M)$

Intersections:

Lemma 13.3. *There exists a sesquilinear form and a commutative diagram*

$$\begin{tikzcd} I_k^0(M) \times I_k^0(M) \arrow[r, "\lambda"] \arrow[d] & \mathbb{Z}\pi \\ H_k(\widetilde{M}) \times H_k(\widetilde{M}) \arrow[r, "\widetilde{\lambda}_{\widetilde{M}}"] & \mathbb{Z}\pi \end{tikzcd}$$

Remark 13.4. $I_k^0(M)$ is a $\mathbb{Z}\pi$-module. $(f,w) \to (f, w \circ \lambda)$

Addition: take along $w_1^{-1} \circ w_0$ λ : make $f_0 \pitchfork f_1$

$$\operatorname{im}(f_0) \cap \operatorname{im}(f_1) = \text{finite set of points} D$$

$X \in D \rightsquigarrow \varepsilon(x)g, \quad g \in \pi_1(M,b)$

Sign: Choose paths in S^q from s to $f_i^{-1}(x)$ Compare orientations using fiber transport. Use the chosen paths in S^q and w_0, w_1 to obtain a loop γ based at b. $g = [\gamma]$. $\Rightarrow f_0 \pitchfork f_1 \rightsquigarrow \sum_{x \in D} \varepsilon(x) g(x)$

Self-intersections: We have after placing in general position, a double point set $D \in \operatorname{im}(f)$

$$x \in D \Rightarrow \varepsilon(x)g(x)$$

with choices of ordering.

$$\mu(f) \in \mathbb{Z}\pi/_{x-\varepsilon\overline{x}}$$

$x \in \mathbb{Z}\pi, \varepsilon = (-1)^q, x = \sum^G n^g g, \overline{x} = \sum^{G'} w(n^g)g^{-1}, w\colon \pi \to \mathbb{Z}_2$ orientation.

Proposition 13.5 (Lueck)**.** *λ, μ define a quadratic form on the $\mathbb{Z}\pi$-module $I_{k^0}(M)$ i.e.*
- $\lambda(\alpha,\beta) = \varepsilon\overline{\lambda(\beta,\alpha)}$
- $\lambda(\alpha, u_1\beta_1 + u_2\beta_2) = u_1\lambda(\alpha_1,\beta_1) + u_2\lambda(\alpha_2,\beta_2)$
- $\mu(\alpha,\beta) = \mu(\alpha) + \mu(\beta) + [\lambda(\alpha,\beta)]$
- $\lambda(\alpha,\beta) = (1+\varepsilon T)\mu(\alpha) \in \mathbb{Z}\pi$
- $\mu(r\alpha) = r\mu(\alpha)\overline{r}$ *where $T\colon \mathbb{Z}\pi \to \mathbb{Z}\pi$ is the involution*

Theorem 13.6 (Wall)**.** *$q \geq 3$.*

$$\mu(\alpha) = 0 \Leftrightarrow \alpha \textit{ is rep. by an embedding } f$$

and

$$\lambda(\alpha,\beta) = 0 \iff \alpha \sim f_0 \textit{ and } \beta \sim f_1 \textit{ with } \operatorname{im}(f_0) \cap \operatorname{im}(f_1) = \emptyset$$

Idea of proof: "Whitney trick"

Local picture IMAGE 4 x_1 and x_2 pair up so that $\varepsilon(x_1) = -\varepsilon(x_2)$ and $g(x_1) = g(x_2)^{-1}$. It follows $\delta_1 \circ \delta_2 \simeq *$ and hence bounds an embedding of D^2. Locally have $D^2 \times D^{2q-2}$ and our "picture" can be shown to be differ(homeo) to $D^2 \times D2q-2 \supseteq D^2 \times (\overbrace{D^{q-1}}^{\subseteq F^1} \times \underbrace{D^{q-1}}_{\subseteq F^2})$. "Now slide"

14. The even dimensional surgery obstruction

Setting: $f\colon M \to X^{2q,}(M, \overline{f}, f)$. Assume f a q-equivalence. $K_q(M) = \pi_{q+1}(f)$

Exercise 14.1. This is a stably free $\mathbb{Z}\pi$-module. It follows $M\#(S^q \times S^q) \to X$. assume $K_q(M)$ is free over $\mathbb{Z}\pi$. Set

$$\sigma(\overline{f}, f) = [K_q(M), \lambda, \mu]$$

where $K_q(M) \xrightarrow{\partial} \pi_q(M) \cong I_q^0(M)$

Here

$$L_{2a:=} \left\{(P, \lambda, \mu) \mid P\text{f.g. free }, \lambda\colon P \xrightarrow{\cong} P^*, \lambda(\mu) \text{ are a quadratic pair}\right\}/_{\simeq}$$

$\simeq$ is generate by isometry. Set metabolic = hyperbolism = 0

Definition 14.2.
- $L \subseteq P$ is a Lagrangian for λ, μ if
  1. L is a half rank summand
  2. $\lambda|_{L\times L} = 0, \mu|_L = 0$
- (P, λ, μ) is metabolic if it contains a Lagrangian

Lemma 14.3. *If* $(P, \lambda, \mu) \overset{i}{\supseteq} (L, 0, 0)$ *is the inclusion of a Lagrangian it extends to an isomorphism*

$$H_\varepsilon(L) \cong (P, \lambda, \mu)$$

$$H_\varepsilon(L) = (L \oplus L^*, \begin{pmatrix} 0 & \mathrm{id} \\ \varepsilon\,\mathrm{id} & 0 \end{pmatrix}, \mu(L) = 0 = \mu, L^*)$$

Proposition 14.4.

$$\sigma(\overline{f}, f) = 0 \iff (\overline{f}, f) \textit{ is normally bordant to a htpy equiv.}$$

Proof. Show that $\sigma(\overline{f}, f)$ is well-defined, depends only on the normal bordism class. Suppose we have a normal bordism

- Do surgery on $W^{2q+1} \xrightarrow{X} X^q$ to make F a q-equiv.
- $H_i(W, M_i; \mathbb{Z}\pi) \cong 0 \iff i = (q, q+1)$ IMAGE 5

It follows by Wall

$$W \cong (M \times I) \cup (\cup q - \text{handles} \cup (q+1) - \text{handles})y <$$

So there exits an intermediate manifold $M_2 \subseteq W$ such that

$$M_2 \cong M_1 \#_r S^q \times S^q$$
$$M_2 \cong M_0 \#_r S^q \times S^q$$

We get

$$(K_q(M_1) \oplus H_\varepsilon(\mathbb{Z}\pi^r, \lambda_1, \mu_1)) \cong (K_q(M_0) \oplus H_\varepsilon(\mathbb{Z}\pi^q), \lambda_0, \mu_0)$$

Choose a basis $\{x_1, \dots (x_k\}) \subset L \subseteq (K_q(M), \lambda, \mu)$. $\{x_i\}$ is represent by $\amalg_i = 1^k D^q \times S^q \hookrightarrow M$. Perform surgery on these embeddings. Check: the outcome is a homotopy equivalence. □

Talk 6

Recall

$$\mathcal{N}(X) \xrightarrow{\sigma} L_n(\mathbb{Z}\pi)$$

$n = 2q \geq 6$

Proposition 14.5.

$$L_0(\mathbb{Z}) \cong 8 \underbrace{\mathbb{Z}(\sigma)}_{signature} \cong \mathbb{Z}(E_8)$$

(Witt group of symmetric bilinear forms)

$$L_2(\mathbb{Z}) \cong \mathbb{Z}_2(A)$$

$$A = (\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix})$$

$n = 2q - 1$

$$\begin{tikzcd} \nu_M \arrow[r, "\bar{f}"] \arrow[d] & \xi \arrow[d] \\ M^n \arrow[r, "f"] & X \end{tikzcd}$$

f a q-equivalence. Take a generating set $x_1, \dots, x_k$ for $K_q(M)$. Represent by $\underbrace{\#_{i=1}^q S^q \times D^{q+1}}_{U} \hookrightarrow M$.

$$M = U \cup M_0 \# S^q \times S^q \to X = D^{2q} \cup X$$

$$H_q(\partial U; \mathbb{Z}\pi) \cong H_\varepsilon(\mathbb{Z}\pi^k)$$

We see two Lagrangians.

$$K \coloneqq \mathrm{im}(H_{q+1}(U, \partial U) \to H_\varepsilon(\mathbb{Z}\pi^k))$$
$$L \coloneqq \mathrm{im}(H_{q+1}(M_0, \partial U) \to H_\varepsilon(\mathbb{Z}\pi^k))$$

Set $\Lambda = \mathbb{Z}\pi$ It follows $H_\varepsilon(\Lambda, K, L)$ a hyperbolic form and two Lagrangians. Check if $K + L = H_\varepsilon(\Lambda)$. Then f is a homotopy equivalence.

$$L_{2q+1}(\mathbb{Z}\pi) \coloneqq \{(H_\varepsilon(\Lambda^r); K, L)\} /_{\simeq}$$

where $(H_\varepsilon(\Lambda^r); K, L)$ is trivial if $H_\varepsilon = K + L$

- have stable (add trivial formations) isometry
- set boundary formations ~ 0

where $H_\varepsilon; K, \Gamma_\theta$ is boundary, Γ_θ = graph of $\theta \colon L \to L^*$ is an $(-\varepsilon)$-quadratic form and we identify $H_\varepsilon = K \oplus K^*$.

Let M

Theorem 14.6.

$$\sigma(\underline{f}, f) = 0 \iff (\underline{f}, f) \sim_{h.b} \textit{ to a h. e.}$$

$$L_{2q+1}(\mathbb{Z}) = 0 \textit{ for all } q$$
$$L_{2q+1}(\mathbb{Z}\pi) = 0 \pi \textit{ of odd order}$$

Theorem 14.7 (Browder-Novikov)**.** *X^n a 1-connected Poincaré complex with reducible SNF, $n \geq 5$. Then*

$$X \simeq M^{2q+1}, smooth n = 2q+1$$
$$X \simeq M^{2q}, PL n = 2q$$
$$X \simeq M^{2q}, smooth$$

if and only if for some reduction ξ of ν_X

$$\sigma_X = \langle \mathscr{L}(-\xi), [X] \rangle$$

Proof. $n = 2q+1$ Apply surgery, $L_{2q+1}(\mathbb{Z}) = 0$
$n = 2q$ $\sigma(M \xrightarrow{f} X) = \sigma(K_q(M, \lambda(\mu) \Rightarrow) H_q)(M) \cong H_q(X) \oplus K_q(M)$ as forms $\Rightarrow \sigma(M \xrightarrow{f} X) = \sigma_M - \sigma_X$. Use bundle data and Hirzebruch σ-theorem. □

15. The surgery exact sequence

Theorem 15.1 (Smale, Barden, Mazur, Stallings)**.** *s/h-co-bordism-theorem. $n+1 \geq 6$. Assume $M_i \to X$ is a homotopy equivalence of Cat-manifolds. W is a h-cobordism from M_0 to M_1*

$$\{W\} / {}_{\textit{Cat-isomorphism rel } M_0} \xrightarrow{\tau} \mathrm{Wh}(\pi) \cong K_1(\mathbb{Z}\pi)/_{\{g\}}$$

then τ is a bijection with

$$\tau'(0) = (M \times I, M \times \{0\}, M \times \{1\})$$

Wh$(e) = 0$ and Wh$(\pi) = 0$ if π is negatively curved. Cat = PL, TOP

Corollary 15.2. *If $n \geq 6$ $\mathscr{S}^{Cat}(S^n) = \{\mathrm{id}\}$*

Proof. Given $\sum^n \simeq S^n$ a Cat structure form $\Sigma^n - (D^n \amalg D^n) = W$ Check W is an h-cobordism. It follows $\Sigma^n \cong D^n \cup_f D^n$. $f\colon S^{n-1} \cong S^{n-1}$. In Cat $=$ PL, TOP we may cone f to get $\Sigma \cong S^n$ □

Definition 15.3. Given M^n a closed Cat manifold form

$$W(M, f_0) = \{W(F\} /_{\simeq) \text{ bordism rel. } M_0 \amalg M_1}$$

where IMAGE5

Theorem 15.4 (Wall realization)**.** *There exist a well-defined bijective surgery obstruction map*

$$\sigma\colon W(M, f_0) \to L_{n+1}(\mathbb{Z}\pi)$$

with $\sigma^{-1}(0) = h$-cobordisms

Proof idea. The surgery obstruction can be defined as before. We do all surgeries in $int(W)$. Suppose $\sigma(W, M_0, M_1; F) = 0$. As before, W is bordant relative $M_0 \amalg M_1$ to a homotopy equivalence. IMAGE 6 We get an h-cobordism W'. □

Aside on simple homotopy equivalences.
There are two structure sets

$$\mathscr{S}^h(M) = \{f\colon N \simeq M\} / {}_{\text{h-cobordism}}$$
$$\mathscr{S}^s(M) = \{f\colon N \simeq M \text{ simple homotopy equivalence }\} / {}_{\text{s-cobordism}}$$

If Wh$(\pi) = 0$ these agree

Theorem 15.5 (B-N-S-W, K-S (TOP)). *There exists a long exact sequence for M^n a closed Cat-manifold, $n \geq 5$*

$$\longrightarrow \underbrace{[\Sigma M_+, G/Cat]}_{\mathcal{N}(M \times I \,\mathrm{rel}\, M \times \partial I)} \longrightarrow L_{n+1}(\mathbb{Z}\pi) \xrightarrow{w} \mathscr{S}^h(M) \xrightarrow{\eta} [M, G/Cat] \xrightarrow{\sigma} L_n(\mathbb{Z}\pi)$$

where

$$\eta(f\colon N \to M) = \left[\begin{array}{ccc} \nu_N & \xrightarrow{\overline{f}} & (f^{-1})^* nu_N \\ \downarrow & & \downarrow \\ N & \xrightarrow{f} & M \end{array}\right]$$

and $\overline{f}$ is well defined up top homotopy.

$$w(x \in L_{n+1}(\mathbb{Z}\pi)) = \partial_+(W, M_0, M_1, F)$$

where $\sigma(W, -) = x$

16. Wall realisation and plumbing

Given $(P, \lambda, \mu) \in L_{2q}(\mathbb{Z}\pi)$ with basis $x_1, \ldots, x_k$ and $M^2q + 1$ IMAGE 7 Attach q-handles to M, one for elements of $\{x_1, \ldots, x_k\}$ along $\phi\colon \amalg_i = 1^k D^q \times S^{q-1} \hookrightarrow M$. If $M = S^2q - 1$ form $W \cup D^2a$. This is a plumbing manifold.

Example 16.1. If $(P, \lambda, \mu) = E_8 \to W^{4k}_{E_8}$ (Milnor manifold) $\partial W_{E_8} = \Sigma^{4k-1} \simeq S^{4k-1}$, M_{E_8} closed PL manifold

W^{4k+2} = Kervaire manifold

M_A = PL-Kervaire manifold.

Example 16.2. $M = S^n$, Cat = PL, TOP, $n \geq 6$

$$\underbrace{\mathscr{S}^{\mathrm{Cat}}(S^n \times I \,\mathrm{rel}\, \partial)}_{=*} \longrightarrow \underbrace{[\Sigma S^n_+, G/\mathrm{Cat}]}_{\pi_{n+1}(G/\mathrm{Cat})} \xrightarrow{\simeq} L_{n+1}(\mathbb{Z}) \xrightarrow{w} \underbrace{\mathscr{S}^{\mathrm{Cat}}(S^n)}_{=*} \xrightarrow{\eta} [S^n, G/\mathrm{Cat}] \xrightarrow{\sigma} L_n(\mathbb{Z})$$

We get

$$\pi_i(G/\mathrm{Cat}) \cong L_i(\mathbb{Z})$$

Example 16.3. $M = T^n$ or more generally an aspherical manifold

Conjecture 16.4 (Borel).

$$\mathscr{S}^{\mathrm{TOP}}(M) = \{[\mathrm{id}_M]\}$$

$[\Sigma M_+, G/\mathrm{Cat}]$ is computable and $L_{n+1}(\mathbb{Z}\pi)$ are mysterious but fundamental. For $M = T^n$ then $\Sigma T^n \simeq \vee \vee S_i \Rightarrow [\Sigma T^n, G/\mathrm{Cat}] \cong \oplus \pi_i(G/\mathrm{Cat}) \cong \oplus L_i(\mathbb{Z}) \Rightarrow L_i(\mathbb{Z}[\mathbb{Z}^n]) \cong \oplus L_{i_j}(\mathbb{Z})$

Talk 6

Recall

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$n = 2q \geq 6$

Proposition 16.5.

$$L_0(\mathbb{Z}) \cong 8 \underbrace{\mathbb{Z}(\sigma)}_{signature} \cong \mathbb{Z}(E_8)$$

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$n = 2q - 1$

$$\begin{tikzcd} \nu_M \arrow[r, "\bar{f}"] \arrow[d] & \xi \arrow[d] \\ M^n \arrow[r, "f"] & X \end{tikzcd}$$

f a q-equivalence. Take a generating set $x_1, \ldots, x_k$ for $K_q(M)$. Represent by $\underbrace{\#_{i=1}^q S^q \times D^{q+1}}_{U} \hookrightarrow M$.

$$M = U \cup M_0 \# S^q \times S^q \to X = D^{2q} \cup X$$
$$H_q(\partial U; \mathbb{Z}\pi) \cong H_\varepsilon(\mathbb{Z}\pi^k)$$

We see two Lagrangians.

$$K := \operatorname{im}(H_{q+1}(U, \partial U) \to H_\varepsilon(\mathbb{Z}\pi^k))$$
$$L := \operatorname{im}(H_{q+1}(M_0, \partial U) \to H_\varepsilon(\mathbb{Z}\pi^k))$$

Set $\Lambda = \mathbb{Z}\pi$ It follows $H_\varepsilon(\Lambda, K, L)$ a hyperbolic form and two Lagrangians. Check if $K + L = H_\varepsilon(\Lambda)$. Then f is a homotopy equivalence.

$$L_{2q+1}(\mathbb{Z}\pi) := \{(H_\varepsilon(\Lambda^r); K, L)\} /_{\simeq}$$

where $(H_\varepsilon(\Lambda^r); K, L)$ is trivial if $H_\varepsilon = K + L$

- have stable (add trivial formations) isometry
- set boundary formations ~ 0

where $H_\varepsilon; K, \Gamma_\theta$ is boundary, Γ_θ = graph of $\theta \colon L \to L^*$ is an $(-\varepsilon)$-quadratic form and we identify $H_\varepsilon = K \oplus K^*$.

Let M

Theorem 16.6.

$$\sigma(\underline{f}, f) = 0 \iff (\underline{f}, f) \sim_{h.b} \textit{ to a h. e.}$$

$$L_{2q+1}(\mathbb{Z}) = 0 \textit{ for all } q$$
$$L_{2q+1}(\mathbb{Z}\pi) = 0 \pi \textit{ of odd order}$$

Theorem 16.7 (Browder-Novikov). *X^n a 1-connected Poincaré complex with reducible SNF, $n \geq 5$. Then*

$$X \simeq M^{2q+1}, \textit{smooth} n = 2q+1$$
$$X \simeq M^{2q}, PL n = 2q$$
$$X \simeq M^{2q}, \textit{smooth}$$

if and only if for some reduction ξ of ν_X

$$\sigma_X = \langle \mathscr{L}(-\xi), [X] \rangle$$

Proof. $n = 2q + 1$ Apply surgery, $L_{2q+1}(\mathbb{Z}) = 0$

$n = 2q$ $\sigma(M \xrightarrow{f} X) = \sigma(K_q(M, \lambda(\mu) \Rightarrow) H_q)(M) \cong H_q(X) \oplus K_q(M)$ as forms $\Rightarrow \sigma(M \xrightarrow{f} X) = \sigma_M - \sigma_X$. Use bundle data and Hirzebruch σ-theorem.

□

17. The surgery exact sequence

Theorem 17.1 (Smale, Barden, Mazur, Stallings)**.** *s/h-co-bordism-theorem. $n+1 \geq 6$. Assume $M_i \to X$ is a homotopy equivalence of Cat-manifolds. W is a h-cobordism from M_0 to M_1*

$$\{W\}/_{\ \textit{Cat-isomorphism rel } M_0} \xrightarrow{\tau} \mathrm{Wh}(\pi) \cong K_1(\mathbb{Z}\pi)/_{\{g\}}$$

then τ is a bijection with

$$\tau'(0) = (M \times I, M \times \{0\}, M \times \{1\})$$

$\mathrm{Wh}(e) = 0$ and $\mathrm{Wh}(\pi) = 0$ if π is negatively curved. Cat = PL, TOP

Corollary 17.2. *If $n \geq 6$ $\mathscr{S}^{Cat}(S^n) = \{\mathrm{id}\}$*

Proof. Given $\sum^n \simeq S^n$ a Cat structure form $\Sigma^n - (D^n \amalg D^n) = W$ Check W is an h-cobordism. It follows $\Sigma^n \cong D^n \cup_f D^n$. $f\colon S^{n-1} \cong S^{n-1}$. In Cat $=$ PL, TOP we may cone f to get $\Sigma \cong S^n$ □

Definition 17.3. Given M^n a closed Cat manifold form

$$W(M, f_0) = \{W(F\}/_{\simeq)\text{ bordism rel. } M_0 \amalg M_1}$$

where IMAGE5

Theorem 17.4 (Wall realization)**.** *There exist a well-defined bijective surgery obstruction map*

$$\sigma\colon W(M, f_0) \to L_{n+1}(\mathbb{Z}\pi)$$

with $\sigma^{-1}(0) = h$-cobordisms

Proof idea. The surgery obstruction can be defined as before. We do all surgeries in *int*(W). Suppose $\sigma(W, M_0, M_1; F) = 0$. As before, W is bordant relative $M_0 \amalg M_1$ to a homotopy equivalence. IMAGE 6 We get an h-cobordism W'. □

Aside on simple homotopy equivalences.
There are two structure sets

$$\mathscr{S}^h(M) = \{f\colon N \simeq M\}/_{\text{ h-cobordism}}$$
$$\mathscr{S}^s(M) = \{f\colon N \simeq M \text{ simple homotopy equivalence }\}/_{\text{ s-cobordism}}$$

If $\mathrm{Wh}(\pi) = 0$ these agree

Theorem 17.5 (B-N-S-W, K-S (TOP))**.** *There exists a long exact sequence for M^n a closed Cat-manifold, $n \geq 5$*

$$\longrightarrow \underbrace{[\Sigma M_+, G/Cat]}_{\mathcal{N}(M\times I \text{ rel } M \times \partial I)} \longrightarrow L_{n+1}(\mathbb{Z}\pi) \xrightarrow{w} \mathscr{S}^h(M) \xrightarrow{\eta} [M, G/Cat] \xrightarrow{\sigma} L_n(\mathbb{Z}\pi)$$

where

$$\eta(f\colon N \to M) = \left[\begin{array}{ccc} \nu_N & \xrightarrow{\overline{f}} & (f^{-1})^* nu_N \\ \downarrow & & \downarrow \\ N & \xrightarrow{f} & M \end{array}\right]$$

and $\overline{f}$ is well defined up top homotopy.

$$w(x \in L_{n+1}(\mathbb{Z}\pi)) = \partial_+(W, M_0, M_1, F)$$

where $\sigma(W, -) = x$

18. Wall realisation and plumbing

Given $(P, \lambda, \mu) \in L_{2q}(\mathbb{Z}\pi)$ with basis $x_1, \ldots, x_k$ and $M^2q + 1$ IMAGE 7 Attach q-handles to M, one for elements of $\{x_1, \ldots, x_k\}$ along $\phi\colon \amalg_i = 1^k D^q \times S^{q-1} \hookrightarrow M$. If $M = S^2q - 1$ form $W \cup D^2a$. This is a plumbing manifold.

Example 18.1. If $(P, \lambda, \mu) = E_8 \to W^{4k}_{E_8}$ (Milnor manifold) $\partial W_{E_8} = \Sigma^{4k-1} \simeq S^{4k-1}$, M_{E_8} closed PL manifold

W^{4k+2} = Kervaire manifold

M_A = PL-Kervaire manifold.

Example 18.2. $M = S^n$, Cat = PL, TOP, $n \geq 6$

$$\underbrace{\mathscr{S}^{\mathrm{Cat}}(S^n \times I \operatorname{rel} \partial)}_{=*} \longrightarrow \underbrace{[\Sigma S^n_+, G/\mathrm{Cat}]}_{\pi_{n+1}(G/\mathrm{Cat})} \xrightarrow{\simeq} L_{n+1}(\mathbb{Z}) \xrightarrow{w} \underbrace{\mathscr{S}^{\mathrm{Cat}}(S^n)}_{=*} \xrightarrow{\eta} [S^n, G/\mathrm{Cat}] \xrightarrow{\sigma} L_n(\mathbb{Z})$$

We get

$$\pi_i(G/\mathrm{Cat}) \cong L_i(\mathbb{Z})$$

Example 18.3. $M = T^n$ or more generally an aspherical manifold

Conjecture 18.4 (Borel)**.**

$$\mathscr{S}^{\mathrm{TOP}}(M) = \{[\mathrm{id}_M]\}$$

$[\Sigma M_+, G/\mathrm{Cat}]$ is computable and $L_{n+1}(\mathbb{Z}\pi)$ are mysterious but fundamental. For $M = T^n$ then $\Sigma T^n \simeq \vee \vee S_i \Rightarrow [\Sigma T^n, G/\mathrm{Cat}] \cong \oplus \pi_i(G/\mathrm{Cat}) \cong \oplus L_i(\mathbb{Z}) \Rightarrow L_i(\mathbb{Z}[\mathbb{Z}^n]) \cong \oplus L_{i_j}(\mathbb{Z})$

Talk 7

19. The TOP-GSES - Introduction

Let X be a Cat n-manifold,$n \geq 5$, $\pi = \pi_1 X$, Cat = DIFF, PL, TOP.

Recall. There are the geometric surgery exact sequence for these categories (CAT-GSES)

$$\longrightarrow \mathcal{N}^{\mathrm{Cat}}(X \times I) \xrightarrow{\theta} L_{n+1}(\mathbb{Z}\pi) \xrightarrow{\partial} \mathscr{S}^{\mathrm{Cat}}(S^n) \xrightarrow{\eta} \mathcal{N}^{\mathrm{Cat}}(X) \xrightarrow{\theta} L_n(\mathbb{Z}\pi) \longrightarrow ???\ .$$

The answers to the uniqueness and existences questions given in previous talks were

A2 (uniqueness). Let $f_0\colon M_0 \xrightarrow{\simeq} X$, $f_1\colon M_1 \xrightarrow{\simeq} X$ be homotopy equivalences.

$$\begin{aligned} \exists h\colon M_0 \xrightarrow{\cong} M_1 \text{ such that } f_1 \circ h \simeq f_0 &\iff [f_0] = [f_1] \in \mathscr{S}^{\mathrm{Cat}}(X) \\ &\iff \eta[f_0] = \eta[f_1] \in \mathcal{N}^{\mathrm{Cat}}(X) \\ &\iff \exists (F, B)\colon (W, M_0, M_1) \to (X \times I, 0, 1) \\ &\qquad \text{such that } \theta(F, B) = 0 \in L_{n+1}(\mathbb{Z}\pi) \end{aligned}$$

A1 (existence). Let X be an n-GPC.

$$\begin{aligned} \exists M^n \text{ Cat n-manifold } \simeq X &\iff \mathscr{S}^{\mathrm{Cat}}(X) \neq \emptyset \\ &\iff \mathcal{N}^{\mathrm{Cat}}(X) \neq 0 \\ &\iff \exists (f, b)\colon M \to X \text{ degree one normal map with} \theta(f, b) = 0 \in L_n(\mathbb{Z}\pi) \end{aligned}$$

Philosophy: Uniqueness is the relative form of existence.

We would like to have something better:

$$\begin{aligned} \text{uniqueness} &\iff \text{"}0\text{"} = s(f_0, f_1) = [f_0] - [f_1] \in \mathscr{S}^{\mathrm{Cat}}(X) \\ \text{existence} &\iff \text{"}0\text{"} = s(X) \in ??? \end{aligned}$$

This would make more sense if $\mathscr{S}^{\mathrm{Cat}}(X) = \pi_{n+1}(\text{ some space })$. Then we could have

$$\begin{aligned} \text{uniqueness} &\in \pi_{n+1} \\ \text{existence} &\in \pi_n \end{aligned}$$

Question. Can we spacify CAT-GSES?

20. Answer for all categories (Quinn-Wall)

The CAT-GSES can be extended to the left ($k \geq 0$)

$$(\text{CAT-GSES})_k \qquad \longrightarrow \mathcal{N}^{\mathrm{Cat}}(X \times D^{k+1}) \xrightarrow{\theta} L_{n+k+1}(\mathbb{Z}\pi) \xrightarrow{\partial} \mathscr{S}^{\mathrm{Cat}}(X \times D^k) \xrightarrow{\eta}$$

$$\mathcal{N}^{\mathrm{Cat}}(X \times D^k) \xrightarrow{\theta} L_{n+k}(\mathbb{Z}\pi) \longrightarrow \ldots$$

where elements in $\mathscr{S}^{\mathrm{Cat}}(X \times D^k)$ are equivalence classes of

$$(f, \partial f) \colon (M, \partial M) \xrightarrow{(\simeq, \cong)} (X \times D^k, X \times S^{k-1})$$

with respect to the equivalence relation

$$(f_0, \partial f_0) \sim (f_1, \partial f_1) \iff \exists h \colon (M_0, \partial M_0) \xrightarrow{\simeq} (M, \partial M_1), f_1 \circ h \simeq f_0 \operatorname{rel} \partial.$$

Theorem 20.1 (Quinn-Wall)**.** *X Cat n-manifold, $n \geq 5$, then there exists a fibration sequence of Δ-sets*

$$\widetilde{\mathscr{S}}^{Cat}(X) \to \widetilde{\mathcal{N}}^{Cat}(X) \to L_n(\mathbb{Z}\pi)$$

such that π_k(CAT-GSES) = (CAT-GSES)$_k$

Idea of proof. $\widetilde{\mathscr{S}}^{\mathrm{Cat}}$ is a pointed Δ-set, i.e. a simplicial set without degeneracies, with k-simplices

$$\widetilde{\mathscr{S}}^{\mathrm{Cat}}(X)^{(k)} = \left\{ (f, \partial f) \colon \underbrace{(M, \partial_i M)}_{(n+k)\text{-dim. CAT manifold } (k+2)\text{-ad}} \xrightarrow{(\simeq, \simeq)} (X \times \Delta^k, X \times \partial_i \Delta^k) \right\}.$$

The face maps ∂_i are given by restriction, the base point is the identity $1 \colon X \times \Delta^k \to X \times \Delta^k$. It has the Kan property which means

$$\pi_k(\widetilde{\mathscr{S}}^{\mathrm{Cat}}(X)) \cong \mathscr{S}^{\mathrm{Cat}}_{\partial}(X \times \Delta^k).$$

Similarly the normal invariants are defined by

$$\widetilde{\mathcal{N}}^{\mathrm{Cat}}(X)^{(k)} = \{ (f(b)) \colon (M, \partial M) \to (X \times \Delta^k, X \times \partial_i \Delta^k) \}$$

with face maps the restriction, base point the identity and Kan property gives

$$\pi_k(\widetilde{\mathcal{N}}^{\mathrm{Cat}}(X)) \cong \mathcal{N}^{\mathrm{Cat}}_{\partial}(X).$$

The L-spaces are little bit harder.

$$\mathscr{L}_n(\mathbb{Z}\pi)^{(k)} = \left\{ (f, b) \colon \underbrace{(M, \partial_i M)}_{(n+k)\text{-dim } (k+3)\text{-ad}} \to \underbrace{(Y, \partial_i Y)}_{(n+k)\text{-dim. } (k+3)\text{-ad}} \text{ degree one normal } \right\}$$

with base point $\emptyset$. We have a additional face with the property

$$[(\partial_{k+2} f_1, \partial_{k+2} b) \colon \partial_{k+2} M \xrightarrow{\simeq} \partial_{k+2} Y]$$

and a reference map $r\colon Y \to K(\pi,1)$. In this case it is harder to show

$$\pi_k(\mathscr{L}^n(\mathbb{Z}\pi)) \cong L_{n+k}(\mathbb{Z}\pi)$$

but it holds. For details see [Wall, §9 and §17A] or [Quinn, geometric formulation of surgery theory]. □

Remark 20.2. (1) The theorem of Quinn-Wall is not quite satisfactory because

$$\mathscr{S}^{\mathrm{Cat}}(X) = \pi_0(\widetilde{\mathscr{S}}^{\mathrm{Cat}}(X))$$

is not a group in general

(2) We would be in a better position if we had spectra. Can we get them?

21. L-spectra

Theorem 21.1 (Quinn-Wall). *We have*

$$\mathscr{L}_n(\mathbb{Z}\pi) \simeq \Omega\mathscr{L}_{n-1}(\mathbb{Z}\pi)$$

Definition 21.2. For $n \in \mathbb{Z}$

$$\mathscr{L}_n(\mathbb{Z}\pi)^{(k)} = \begin{cases} \text{as before} & (n+k) \geq 0, \\ \{\emptyset\} & (n+k) < 0 \end{cases}$$

Corollary 21.3. *The spaces $\mathscr{L}_{-n}(\mathbb{Z}\pi)$ form a spectrum $\mathscr{L}_\bullet(\mathbb{Z}\pi)$ with*

$$\pi_k(\mathscr{L}_\bullet(\mathbb{Z}\pi)) = L_k(\mathbb{Z}\pi)$$

Idea of proof. We have to compare the k-simplices of □

$$\mathscr{L}_n(\mathbb{Z}\pi)^{(k)} \text{ and } \Omega\mathscr{L}_{n-1}(\mathbb{Z}\pi)^{(k)} \subset \mathscr{L}_{n-1}(\mathbb{Z}\pi)^{(k+1)}$$

"Optical Illusion":

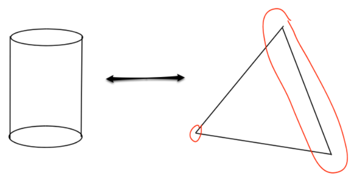

FIGURE 1. $(n+k)$-dim $(k+2)$-ad $\longleftrightarrow$ $n-1+(k+1)$-dimensional $(k+3)$-ad with $\partial_0 \ldots \partial_{k+1} = \emptyset$ and $\partial_{k+3} = \emptyset$

Question. What about $\mathscr{S}^{\mathrm{Cat}}(X), \mathcal{N}^{\mathrm{Cat}}(X)$?

Recall.

$$\mathcal{N}^{\mathrm{Cat}}(X) \cong [X; G/\mathrm{Cat}]$$

is group via the H-spaces structure of $G/\mathrm{Cat}^{\oplus}$ coming from the Whitney sum.

Proposition/Exercise 21.4. *The surgery obstruction map*

$$\theta\colon \mathcal{N}^{Cat}(X) = [X; G/Cat^{\oplus}] \to L_n(\mathbb{Z}\pi)$$

is not a homomorphism in general

22. Answer for Cat = TOP

Theorem 22.1 (Poincaré conjecture)**.**

$$\widetilde{\mathscr{S}}^{\mathrm{TOP}}(\mathrm{pt}) \simeq *$$

Proof.

$$\pi_n \widetilde{\mathscr{S}}^{\mathrm{TOP}}(\mathrm{pt}) = \mathscr{S}_\partial^{\mathrm{TOP}}(D^n) = \mathscr{S}^{\mathrm{TOP}}(S^n) = \{1\}$$

□

Corollary 22.2. *The surgery obstruction map induces an homotopy equivalence*

$$\eta \colon \mathbb{Z} \times \mathrm{G/TOP} \xrightarrow{\simeq} \mathscr{L}_0(\mathbb{Z}).$$

Corollary 22.3. *There exists a H-space structure on* G/TOP *such that θ is a homomorphism.*

Corollary 22.4. *There exists a fibration sequence of spectra*

(TOP-GSFS-spectra) $$\widetilde{\mathscr{S}}_\bullet^{\mathrm{TOP}}(X) \longrightarrow \widetilde{\mathcal{N}}_\bullet^{\mathrm{TOP}}(X) \longrightarrow \mathscr{L}_{\bullet+n}(\mathbb{Z}\pi)$$

such that π_k (TOP-GSFS-spectra) = (TOP-GSES)$_k$

$$\mathscr{S}^{\mathrm{TOP}}(X) \longrightarrow \mathcal{N}^{\mathrm{TOP}}(X) \longrightarrow L_n(\mathbb{Z}\pi) \longrightarrow \underbrace{\pi_1 \mathscr{S}_\bullet^{\mathrm{TOP}}(X)}_{=???} \longrightarrow \pi_1 \mathcal{N}_\bullet^{\mathrm{TOP}}(X)$$

Aim. For all X locally finite simplicial complex define a fibration sequence of spectra

$$\underbrace{\mathbb{S}_\bullet(X)}_{\text{mixture}} \to \underbrace{\mathbb{L}_\bullet\langle 1\rangle(\mathbb{Z}_*(X))}_{\text{local}} \to \underbrace{\mathbb{L}_\bullet(\mathbb{Z}\pi)}_{\text{global}}$$

Comment: $\widetilde{\mathcal{N}}^{\mathrm{TOP}}(X) \simeq \mathrm{Map}(X, \mathrm{G/TOP})$

such that if X is a n-manifold

$$\begin{tikzcd}
{} \arrow[r] & L_{n+1}(\mathbb{Z}\pi) \arrow[r] \arrow[d, "="] & \mathscr{S}^{TOP} \arrow[r] \arrow[d, "\cong"] & \mathcal{N}^{\mathrm{TOP}}(X) \arrow[r, "\sigma"] \arrow[d, "\cong"] & L_n(\mathbb{Z}\pi) \arrow[d, "="] & \\
{} \arrow[r] & L_{n+1}(\mathbb{Z}\pi) \arrow[r] & \mathbb{S}_{n+1}(X) \arrow[r] & H_n(X; \mathbb{L}_\bullet\langle 1\rangle) \arrow[r] & L_n(\mathbb{Z}\pi) \arrow[r] & \mathbb{S}_n(X)
\end{tikzcd}$$

Talk 8

23. The unreasonable effectiveness of cobordism in the homotopy theory of manifolds

Ω_n = cobordism of n-manifolds.

1. (W, M, M') s-cobordism implies $(W; M, M') \cong_{\text{diffeo}} M \times ([0,1], \{0\}, \{1\})$.
   $\Rightarrow M \xrightarrow{\simeq} W \xrightarrow{\simeq} M'$ is homotopic to a diffeomorphism.
2. What about an algebraic analogue of the s-cobordism theorem?

n-dimensional n-manifolds $\to$ chain complexes C with n-dimensional Poincaré duality $\varphi \colon C^{n-*} \xrightarrow{\simeq} C$

$M^n \mapsto (C(M), \varphi)$ with $\varphi_0(M) \cap - \colon C(M)^{n-*} \to C(M), \varphi_1 \colon \varphi_0 \simeq \varphi_0^*, \varphi_2 \colon \varphi_1 \simeq \varphi_1^*$

IMAGE 1

$$\begin{tikzcd}
D^{n-*} \arrow[r, "f^*"] \arrow[dr, "f'^*"'] & C^{n-*} \arrow[r, "\varphi_0"] & C \arrow[r, "f"] & D \\
& C'^{n-*} \arrow[r, "\varphi'_0"] & C' \arrow[ur, "f'"'] &
\end{tikzcd}$$

$$\delta\varphi : f\varphi_0 f^* \simeq f'\varphi_0' f'^* : D^{n-*} \to D$$

$$[M] - [M'] = \partial[W]$$

Poincaré -Lefschetz duality is a chain equivalence

$$\begin{pmatrix} \delta\varphi \\ f\varphi_0 f^* \\ f'\varphi_0' f'^* \end{pmatrix} : D^{n+1-*} \xrightarrow{\simeq} \mathscr{C}((f\ f') : C \oplus C' \to D)$$

algebraic cobordism of n-dimensional symmetric Poincaré complexes (C, φ) over $\mathbb{Z}$ (in the first instance) with cobordism group $L^n(\mathbb{Z})$

The function

$$: \Omega_n \longrightarrow L^n(\mathbb{Z}),$$
$$M \mapsto (C(M), \varphi)$$

is a surjection of abelian groups with

Theorem 23.1. *Two symmetric Poincaré complexes* (C, φ) *,*(C', φ') *are chain homotopy equivalent (i.e. there exists chain equivalence* $h : C \xrightarrow{\simeq} C'$ *such that* $\delta\varphi : h\varphi h^* \simeq \varphi' : C'^{n-*} \to C'$*) if and only if there exist an algebraic* h*-cobordism*

$$((f\ f') : C \oplus C' \to D, (\delta\varphi, \varphi \oplus -\varphi'))$$

i.e. f *and* f' *are chain equivalences.*

G-hypercohomology and G-hyperhomology with $G = \mathbb{Z}_2$. Let V be a $\mathbb{Z}[\mathbb{Z}_2]$-module with involution $T : V \curvearrowright$, $T^2 = \text{id}$. $\mathbb{Z}_2$-cohomology

$$H^0(\mathbb{Z}_2; V) = \{v \in V \mid Tv = v\} = \text{ fixed points of T } = \ker(1 - T : V \to V)$$

$\mathbb{Z}_2$-homology

$$H_0(\mathbb{Z}_2, V) = \text{coker}(1 - T : V \to V) = \text{ orbit of } T = V / \{v \sim Tv\}$$

$\mathbb{Z}_2$-hypercohomology of a $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex V

$$H^n(\mathbb{Z}_2; V) = H_n(\text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, V))$$

with W the standard free $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of $\mathbb{Z}$ with $T = 1$.

$v \in H^n(\mathbb{Z}; V) = \{[v_0, v_1, \ldots]\}$

$v : W \to V_{*+n}\mathbb{Z}[\mathbb{Z}_2]$-module chain map up to chain homotopy

IMAGE2

$$f \in \text{Hom}^{\mathbb{Z}[\mathbb{Z}_2]}(W, V)_n \sum_{q-p=n} \text{Hom}(W_p, V_q)$$

$$d_{\text{Hom}}(f) = d_V f \pm f d_W$$

$$H^n(\mathbb{Z}_2; V) = H_n(\text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, V)) \text{ } \mathbb{Z}_2\text{-hyperhomology}$$

$$H_n(\mathbb{Z}_2; V) = H_n(W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} V) \text{ } \mathbb{Z}_2\text{-hypercohomology}$$

$$\longrightarrow H_n(\mathbb{Z}_2, V) \xrightarrow{1+T} H^n(\mathbb{Z}_2, V) \longrightarrow \widehat{H}^n(\mathbb{Z}_2, V) \longrightarrow H_{n-1}(\mathbb{Z}_2, V)$$

$(W = C(S^\infty = B\mathbb{Z}))$

How do $H^n(\mathbb{Z}_2, V)$ and $H_n(\mathbb{Z}_2, V)$ apply in topology?

For any space X define the $\mathbb{Z}_2$-space $X \times X \curvearrowright T$ $(T(x(y) =)((\text{y,x})))$ and $V = C(X \times X)$

Definition 23.2.

$$Q^n(C(X)) = H^n(\mathbb{Z}_2, C(X \times X)) \overset{?}{=} H^n(\mathbb{Z}_2, C(X) \times C(X))$$
$$Q_n(C(X)) = H_n(\mathbb{Z}_2, C(X \times X)) \overset{?}{=} H_n(\mathbb{Z}_2, C(X) \times C(X))$$

$(T(x \otimes y) = \pm y \otimes x)$

The Eilenberg-Zilber theorem gives a chain equivalence

$$\begin{tikzcd}
C(X) \arrow[r, "\Delta"] \arrow[d, "T=\mathrm{id}"] & C(X \times X) \arrow[r, "\simeq t_0"] \arrow[d, "T"] \arrow[dr, "t_1"] & C(X) \otimes C(X) \arrow[d] \\
C(X) \arrow[r, "\Delta"] & C(X \times X) \arrow[r, "\simeq \overline{t_0}"] & C(X) \otimes C(X)
\end{tikzcd}$$

$$E_1 : E_0 T \simeq T E_0$$
$$E_2 : E_1 T \simeq T E_1$$
$$\dots$$

$$\cup \colon H^p(X) \otimes H^q(X) \longrightarrow H^{p+q}(X),$$
$$x \otimes y \mapsto E_0(x \otimes y) = x \cup y$$

Cup product only commute on cohomology $x \cup y = \pm y \cup x$ NOT on the chain level.

$$\{E_s\} \in H_0(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_{\mathbb{Z}}(C(X \times X), C(X) \otimes C(X))))$$
$$E_0 \colon C(X \times X) \to C(X) \otimes C(X)$$

is a $\mathbb{Z}$-chain equivalence which fails to be $\mathbb{Z}_2$-equivariant up to higher homotopy.

Definition 23.3. The symmetric structure groups of a $\mathbb{Z}$-module chain complex C are

$$Q^n(C) = H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes C))$$

Definition 23.4. The quadratic structure groups of a $\mathbb{Z}$-module chain complex C are

$$Q_n(C) = H_n((W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} C \otimes C))$$

For any space X there is a natural transformation

$$\varphi \colon H_n(X) \to Q^n(C(X))$$
$$C(X) \to \mathrm{Hom}(W, C(X) \otimes C(X))$$

for any f.g. free $\mathbb{Z}$-module K define

$$C \colon \dots \to 0 \to K^* = \mathrm{Hom}_{\mathbb{Z}}(K; \mathbb{Z}) \to 0 \to \dots$$
$$\varphi \in Q^{2k}(C) = \left\{\varphi \in \mathrm{Hom}_{\mathbb{Z}}(K, K^*) \mid \varphi^* = (-)^i \varphi\right\}$$
$$\mathrm{Hom}_{\mathbb{Z}}(K, K^*) \cong K^* \otimes_{\mathbb{Z}} K^*$$

Talk 9

24. Zoo of structures on chain complexes

Let A be a ring with involution.

Symmetric L-theory.

$$Q^n(C) = H_n(Hom_{\mathbb{Z}[\mathbb{Z}_2]})(W, C \otimes_A C) = \left\{ \begin{array}{c} \text{n-dimensional symmetric} \\ \text{strucutres on a} \\ \text{A-module chain complex} \end{array} \right\}$$

symmetric Q-group. The Alexander-Whitney-Steenrod symmetric construction for any space X with universal cover $\widetilde{X}$

$$\varphi_X \colon H_n(X) \longrightarrow Q^n(C(\widetilde{X})),$$
$$[X] \mapsto \varphi_X$$
$$\varphi_X[X]_0 = [X] \cap - \colon C(\widetilde{X})^{n-*} \to C(\widetilde{X})$$

(cap product)

$$L^n(A) = \left\{ \begin{array}{c} \text{cobordism classes of } (C, \varphi) \text{ for} \\ \text{fixed A with } \varphi_0 \colon C^{n-*} \to C \\ \text{an A-module chain equivalence} \end{array} \right\}$$

If X is an n-dimensional GPC then $(C(\widetilde{X}), \varphi(X))$ is an n-dimensional SAPC

$$\sigma \colon \Omega(K) \longrightarrow L^n(\mathbb{Z}[\pi(K)]),$$
$$M^n \to K \mapsto C(\widetilde{M}), \varphi_M(M)$$

For a space X with $w \colon \pi_1 X \to \mathbb{Z}_2 = \{\pm 1\}$. Let $A = \mathbb{Z}[\pi_1 X]$ with involution $\overline{g} = w(g)g^{-1} \in Z[\pi_1 X]$ for $g \in \pi_1(X)$.

The symmetric theory was initiated by Mishchenko (1974). Generalization of signature (If $n = 4k$ then $\sigma^*(M) \in L^{4k}(\mathbb{Z}[pi_1(M)]) \to L^{4k}(\mathbb{Z}) = \mathbb{Z}$ is the signature)

$$\Omega_n(K) = \underbrace{H_n(K, \text{MSO})}_{\text{generalized homology}} \xrightarrow{A} \underbrace{L_n(\mathbb{Z}[pi_1(K)])}_{\text{not homology in general}}$$

Quadratic L-theory. (Wall 1969)

$$\begin{aligned} Q_n(C) &= H_n(W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} C \otimes_A C) = H_n(\mathbb{Z}_2; C \otimes_A C) \\ &= \{ \text{ quadratic structure on A-module chain complexes } C \text{ } \} \end{aligned}$$

$$\begin{array}{ll} \varphi = \{\varphi_s \in (C \otimes_A C)_{n+s} \mid s \geq 0\} & \psi = \{\psi_s \in (C \otimes_A C)_{n-s} \mid s \geq 0\} \\ d\varphi_s = \varphi_{s-1} \pm T f_{s-1} & d\psi_s = \psi_{s+1} \pm T\psi_{s+1} \end{array}$$

$$(1 + T)\psi \leftarrow\!\shortmid \psi$$

Theorem 24.1.

$$L_n(A) = \left\{ \begin{array}{c} \textit{cobordism groups of } n\textit{-dimensional qudratic} \\ \textit{Poincaré complexes } (C, \psi) \textit{ over } A \textit{ with} \\ (1 + T)\psi \colon C^{n-*} \to C \textit{ a chain equivalence} \end{array} \right\}$$

Identify $L_n(A)$ with the subgroup of (C, ψ) such that

$C \colon \ldots \to 0 \to C_n \to 0 \to \ldots$ if $n = 2k$ $\quad H_i(C) = 0$ for $i \neq k$

$C \colon \ldots \to 0 \to C_{n+1} \to 0 \to \ldots$ if $n = 2k + 1$ $\quad H_i(C) = 0 \neq k + 1$

How does a normal map $(\overline{f}, f) \colon M \to X$ determine an n-dimensional QAPC (C, ψ) over $A = \mathbb{Z}[\pi_1(X)]$ such that

$$\sigma(f) \in L_n(\mathbb{Z}[\pi_1(X)]) \xrightarrow{\simeq} (C, \psi) \in L_n(\mathbb{Z}[\pi_1(X)])$$

$$\mathrm{Th}(\overline{f})\colon \mathrm{Th}(\nu_M) \to \mathrm{Th}(\xi)$$

map of Thom spaces.

S-dual

$$F = T(\overline{f})\colon T(\xi)^* \simeq X_+ \to T(\nu^m)^* \simeq M_+$$

inducing the Umkehr map

$$f'\colon C(X) \simeq C(X)^{n-*} \xrightarrow{f^*} C(M)^{n-*} \simeq C(M)$$
$$F\colon \Sigma^\infty X_+ \to \Sigma^\infty M_+$$

(There is also $\pi_1 X$ equivariant version)

$$\begin{tikzcd}
H_n(X) \cong H_{n+\infty}(\Sigma^\infty X_+) \arrow[r, "\varphi_{\Sigma^\infty X_+}"] \arrow[d, "F"] \arrow[rr, bend left, "\varphi_X"] & Q^{n+\infty}(C(\Sigma^\infty X_+)) \arrow[d, "F\otimes F"] & Q^n(C(X)) \arrow[l, "S^\infty"'] \arrow[d, "f'\otimes f'"] \\
H_n(M) \cong H_{n+\infty}(\Sigma^\infty M_+) \arrow[r, "\varphi_{\Sigma^\infty M_+}"] \arrow[rr, bend right, "\varphi_M"'] & Q^{n+\infty}(C(\Sigma^\infty M_+)) & Q^n(C(M)) \arrow[l, "S^\infty"]
\end{tikzcd}$$

$$\varphi_M F - (R \otimes F)\varphi_X$$

$$S\colon Q^n(C) \longrightarrow Q^{n+1}(SC),$$
$$\{\varphi_s \mid s \geq 0\} \mapsto (S\varphi)_s = \varphi_{s-1} \mid s \geq 0$$

Cup products are 0 in suspension

$$\begin{tikzcd}
H_n(X) \otimes H_n(Y) \arrow[r, "\varphi_X \otimes \varphi_Y"] \arrow[d] & Q^n(C(X) \otimes Q^n(C(Y)) \arrow[d] \\
H_n(X \otimes Y) \arrow[r, "\varphi_{X\times X}"] & Q^{n+m}(C(X) \otimes C(Y)) = Q^{m+n}(C(X \times Y))
\end{tikzcd}$$

$$\begin{tikzcd}
H_n(X) \otimes H_n(S^1) \arrow[r, "\varphi_X \otimes \varphi_Y"] \arrow[d] & Q^n(C(X) \otimes Q^n(C(S^1)) \arrow[d] \\
H_n(X \otimes S^1) \arrow[r, "\varphi_{X\times X}"] & Q^{n+m}(C(X) \otimes C(Y)) = Q^{m+n}(C(X \times Y))
\end{tikzcd}$$

[...]

Quadratic construction. The quadratic construction for any ∞-stable map $F\colon \Sigma^\infty X_+ \to \Sigma^\infty M_+$

$$\psi_F\colon H_n(X) \to Q_n(C(M)) \xrightarrow{1+T} Q^n(C(M))$$
$$\psi = \{\psi_s C(M) \otimes C(M))_{n-s} \mid s \geq 0\}$$
$$((1+T)\psi)_s = \begin{cases} (1+T)\psi_0 & s = 0 \\ 0 & s \geq 1 \end{cases}$$
$$d(\psi_s) = \psi_{s+1} \pm T\psi_{s+1}$$

The stable map

$$F\colon \Sigma^\infty X_+ \to \Sigma^\infty M_+$$

induces a chain map

$$f^!\colon C(X) \to C(M)$$

an

$$(1+T)\psi^F = \varphi_M f^! - (f^! \otimes f^!)\varphi_X$$

(No $X \to M$ in general)

$$\begin{tikzcd}
H_n(X) \arrow[rr, "\varphi_X"] \arrow[dd, "f^!"'] \arrow[dr, "\psi"] & & Q^n(C(X)) \arrow[dd, "f^! \otimes f^!"] \\
& Q_n(C(M)) \arrow[dr, "1+T"] & \\
H_n(M) \arrow[rr, "\varphi_M"] & & Q^n(C(M))
\end{tikzcd}$$

For a normal map $(\overline{f}, f)\colon M^n \to X$ have S-dual

$$F = \mathrm{Th}(\overline{f})^* \, \mathrm{Th}(\xi)^* = \Sigma^\infty X_+ \to \mathrm{Th}(\nu^M)^* = \Sigma^\infty M_+$$

inducing $f^!\colon C(X) \simeq C(X)^{n-*} \xrightarrow{f^*} C(M)^{n-*} \simeq C(M)$. Let

$$C(X) \xrightarrow{f^!} C(M) \xrightarrow{e} \mathscr{C}(f^!)$$

be the algebraic mapping cone

Theorem 24.2.

$$H_*(\mathscr{C}(f^!)) = K_*(M) = \ker(f^*\colon) H_*(M) \to H_*(X)$$

$$\mathscr{C}(f^!), \psi = (e \otimes e)\psi_F[X]$$

is an n-dimensional QAPC representing the Wall surgery obstruction

$$L_n^{Wall}(\mathbb{Z}[\pi_1(X)]) \xrightarrow{\simeq} L_n^{QAPC}(\mathbb{Z}[\pi_1(X)])$$

$$\textit{Wall obstruction } \sigma(\overline{f}(f)) \mapsto \ \textit{QAPC obstruction } (\mathscr{C}(f^!), \psi)$$

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$$\begin{tikzcd}
C(X \otimes Y) \arrow[rr, "\varphi_{X\times Y}"] \arrow[d, "\simeq"] \arrow[drr, dashed, "\psi"] & & \mathrm{Hom}(W, (C(X \times Y) \otimes C(X \times Y))) \arrow[d, "\simeq"] \\
C(X) \otimes C(Y) \arrow[dr, "\varphi_X \otimes \varphi_Y"] & & \mathrm{Hom}(W, (C(X) \otimes C(Y)) \otimes (C(X) \otimes C(Y))) \\
& \mathrm{Hom}(W, C(X) \otimes C(X)) \otimes \mathrm{Hom}(W, C(Y) \otimes C(Y)) \arrow[ur, "\Delta^* W"] &
\end{tikzcd}$$

In particual this is the case for $Y = S^1$

$$\varphi_{\Sigma^\infty X_+} \simeq \varphi_{X_+} \otimes \varphi_{S^1}$$

25. Zoo continued

$$\longrightarrow Q_n(C) \xrightarrow{1+T} \widehat{Q}^n(C) = \widehat{H}^n(\mathbb{Z}_2; C \otimes_A C) \longrightarrow Q_{n-1}(C)$$

$$\widehat{\varphi} = \{\widehat{\varphi}_s \in (C \otimes_A C)_{n+s} \mid s \in \mathbb{Z}\} \in \widehat{Q}^n(C)$$

$$\widehat{Q}^0(\mathbb{Z}) = \mathbb{Z}_2$$

$$\Rightarrow \widehat{Q}^n(C) \text{ is a } \mathbb{Z}_2\text{-module}$$

$2\widehat{\varphi} = 0 \in \widehat{Q}^n(C)$ for all $\widehat{\varphi} \in \widehat{Q}^n(C)$

Definition 25.1.

$$S \colon Q^n(C) \to Q^{n+1}(SC), \varphi \mapsto S\varphi$$

$$(S\varphi)_s = \begin{cases} 0 & s = 0 \\ \varphi_{s-1} & s \geq 1 \end{cases}$$

Theorem 25.2.

$$\widehat{Q}^n(C) = \lim_k Q^{n+k}(S^k C) = \lim(Q^n(C) \xrightarrow{S} Q^{n+1}(SC) \xrightarrow{S} \to \ldots)$$

for bounded C. *So* $\varphi \in \operatorname{im}(1 + T \colon Q_n(C) \to Q^n(M))$ *if and only if* $S^k\varphi = 0 \in Q^{n+k}(S^k C)$ *for some* $k \geq 0$

$$W \otimes C \otimes_A C \simeq \lim_k (\operatorname{Hom}(W, C \otimes_A C) \xrightarrow{S^k} S^k \operatorname{Hom}(W, S^k \otimes S^k C)), \varphi \mapsto S^k\varphi = \partial\chi$$

$$\begin{array}{ccccc} H_n(C \otimes_A C) & \longrightarrow & Q^n(C) & \xrightarrow{C} & Q^{n+1}(SC) \\ \psi \mapsto (1+T)\psi & & & & S\varphi \end{array}$$

Claim: For $F \colon \Sigma^\infty X_+ \to \Sigma^\infty M_+$

$$\begin{tikzcd} H_n(X) \arrow[d, "\psi_F"'] \arrow[dr, "\varphi_M F - (F \otimes F)\varphi_X"] \arrow[drr, "?0"] & & \\ Q_n(C(M)) \arrow[r, "1+T"] & Q^n(C(M)) \arrow[r, "J"] & \widehat{Q}^n(C(M)) \end{tikzcd}$$

$J(\varphi) = \partial X$

Hyperquadratic construction. Given a pointed finite CW complex X what is the topoogical meaning of

$$H_n(X) \xrightarrow{\varphi_X} \to Q^n(C(X)) \xrightarrow{J} \widehat{Q}^n(C(X))$$

It depends on the hyperquadratic construction on an S-dual $Y = X^*$ of X (which for an GPC X is the Thom space of the SNF)

The hyperquadratic construction on a space Y with an S-dual $S^N \to Y \wedge Y^*$ such that

$$H_*(Y) \cong H^{N-*}, Y^*$$

is

$$\begin{tikzcd} \theta_Y \colon H^k(Y) \arrow[rr] \arrow[d, "\simeq"] & & \widehat{Q}^{-k}(C(Y)^{-*}) \arrow[d, "="] \\ H_{N-k}(Y^*) \arrow[r, "\varphi_{Y^*}"] & Q^{N-k}(C(Y^*)) \arrow[r, "J"] & \widehat{Q}^{N-k}(\underbrace{C(Y)^{N-*}}_{\simeq C(Y^*)}) \end{tikzcd}$$

Symmetric construction on S-dual

$$J\varphi_X = \theta_{X^* = \text{ S-dual of } X}$$

Let $(\overline{f}, f) \colon M \to X$ be an n-dimensional normal map.

$$\operatorname{Th}(\overline{f}) \colon \operatorname{Th}(\nu_M) \to \operatorname{Th}(\xi)$$

$$\operatorname{Th}(\overline{f})^* = F \colon \operatorname{Th}(\xi)^* = \Sigma^\infty X_+ \to \operatorname{Th}(\nu_M)^* = \Sigma^\infty M_+$$

Let $\nu_M\colon M \to \mathrm{BO}(k)$

$$\begin{tikzcd}
H_n(X) \arrow[r, "="] \arrow[d, "\varphi_X"] \arrow[rr, bend left, "f^!=F"] & H^k(\mathrm{Th}(\xi)) \arrow[r, "\mathrm{Th}(\overline{f})"] \arrow[d, "\theta_{\mathrm{Th}(\xi)}"] & H_n(M) \cong H^k(\mathrm{Th}(\nu_M)) \arrow[d, "\theta_{\mathrm{Th}(\nu_M)}"] \arrow[ddr, "\varphi_M"] & \\
Q^n(C(X)) \arrow[dr, "J"] & \widehat{Q}^k(C(\mathrm{Th}(\xi))^? \arrow[r, "\widehat{f}^*\otimes\widehat{f}^*"] & \widehat{Q} = Q^{n+k}(SC(M)) & \\
 & \widehat{Q}^n(C(X)) \arrow[u, "\cong"] \arrow[rr, "F\otimes F"] & & Q^n(C(M))
\end{tikzcd}$$

[...]

A GPC X has SNF $(\nu_X\colon X \to \mathrm{BG}(k), \rho_X\colon S_{n+k} \to \mathrm{Th}(\nu^x))$ $[X] \in H_n(X) \cong H^k(\mathrm{Th}(\nu_X)) \ni u_{\nu_X}$

$$(C(X), \varphi_X[X] \in Q^n(C(X)))$$

$$J\varphi_X[X] = \theta_{\mathrm{Th}(\nu_X)}(\text{ Thom class } u_{\nu_X})$$

A $(k-1)$-spherical fibration $\nu\colon X \to \mathrm{BG}(k)$ has a canonical hyperquadatic complex

$$\begin{tikzcd}
\widehat{\sigma}^*(\nu) = (C(X), \theta(\nu) \in \widehat{Q}^0(C(X))^{-*}) \\
\text{thom class } u_{\nu_X} \in H^k(\mathrm{Th}(\nu_X)) \arrow[u, "\theta_{\mathrm{Th}(\nu_X)}"]
\end{tikzcd}$$

$$\theta(\nu) = \theta_{\mathrm{Th}(\nu)}(u_\nu)$$

If X is an n-dimensional GPC with symmetric complex $\sigma^*(X) = C(X), \varphi_X[X]$ then if $\nu_X\colon X \to BK(k)$ is SNF

$$J\sigma^*(X) = \widehat{\sigma}^*(\nu_X)$$

Formula of Wu and Thom relation $Sq\colon H^i(X) \to H^*(X)$ to $w(\nu_X)$ What about $\rho_X\colon S^{n+k} \to \mathrm{Th}(\nu_X)$?

Talk 11.1

26. Additve Categories with chain duality

Recall (Lecture 8- 10). Let R be a ring with involution. We have defined symmetric and quadratic L-groups $L^n(R), L_n(R)$ via chain complexes. For $R = \mathbb{Z}\pi$ we have symmetric and quadratic signatures

$$\mathrm{sign}^{L_\bullet}_{\mathbb{Z}\pi}(f,b) := (\mathscr{C}(f^!), (e\otimes e)\psi_F([X])) \in L^n(\mathbb{Z}\pi)$$
$$\mathrm{sign}^{L^\bullet}_{\mathbb{Z}\pi} := [(C(M), \varphi_M([M]))] \in L_n(\mathbb{Z}\pi)$$

where M manifold,$(f,b)\colon M \to X$ degree one normal map and $f^!\colon C(\widetilde{X}) \to C(\widetilde{M}) \simeq \mathscr{C}(f^!)$ the Umkehr map.

Need $L_n($ additive category$)$?
Idea.

- $\mathbb{A}$ = additive category
- $\mathbb{B}(\mathbb{A})$ = bounded chain complexes in $\mathbb{A}$

What is $\mathrm{Hom}_{\mathbb{Z}_2}(W, C \otimes_A C)$ for $C \in \mathbb{B}(\mathbb{A})$.

For $\mathbb{A} = R$-modules

$$- \setminus - \colon C \otimes_R D \longrightarrow \mathrm{Hom}_R(\underbrace{C^{-*}}_{TC}, TC),$$

$$x \otimes y \xrightarrow{\cong} f \mapsto \overline{f(x)}y$$

Definition 26.1. For $T \colon \mathbb{A}^{op} \to \mathbb{B}(\mathbb{A})$ define $T \colon \mathbb{B}(\mathbb{A})^{\mathrm{op}} \to (\mathbb{B}, \mathbb{A})$ by

$$T(C) = \mathrm{Tot}(T, C_p)_q$$

Definition 26.2. Let $\mathbb{A}$ be an additive category. A chain duality on $\mathbb{A}$ is (T, e)

- $T \colon \mathbb{A}^{op} \to \mathbb{B}(\mathbb{A})$
- $e \colon T^2 \xrightarrow{\simeq} 1$ such that $\forall M \in A$
  (1) $e_M \colon T^2(M) \xrightarrow{\simeq} M$
  (2) $T(M) \xrightarrow{T_{e_M}} T^2(M) \xrightarrow{e}_{T(M)} T(M)$

Definition 26.3. $M, N \in \mathbb{A}$

$$M \otimes_{\mathbb{A}} N \coloneqq \mathrm{Hom}_{\mathbb{A}}(TM, N) C \otimes_{\mathbb{A}} D \coloneqq \mathrm{Hom}^{\mathbb{A}}(TC, D)$$

chain complexes of abelian groups.

$$T_{M,N} \colon M \otimes_{\mathbb{A}} N = \mathrm{Hom}_{\mathbb{A}}(TM, N) \to N \otimes_{\mathbb{A}} M = \mathrm{Hom}_{\mathbb{A}}(TN, M)$$

$$TM \xrightarrow{\varphi} NT(N) \xrightarrow{T} (\varphi)T^2(M) \xrightarrow{e_M} M$$

$T_{C,D} \colon C \otimes_{\mathbb{A}} D \to D \otimes_{\mathbb{A}} C,$

$(T_{C,D})_{p,q} = (-1)^{pq} T_{C_p, D_q}$

Definition/Exercise/Proposition 26.4.

- We can define $\widehat{Q}^*_*(C)$ for $C \in \mathbb{B}(\mathbb{A})$.
- SAPC, QAPC $\varphi_0 \colon \Sigma^n(TC) \xrightarrow{\simeq} C$
- $L_n(\mathbb{A}), L^n(\mathbb{A})$ are cobordism groups of n-dimensional QAPCs, SAPCs

Functoriality.

Definition 26.5. A functor of additive categories with chain duality is an additive functor $F \colon \mathbb{A} \to AA'$ such that

- $\forall A \in \mathbb{A} G(A) \colon T'F(A) \xrightarrow{\simeq} FT(A)$ natural map
- $\forall A \in \mathbb{A}$

$$\begin{tikzcd}
T'FT(A) \arrow[r, "G(T(A))"] \arrow[d, "T'(G(A))"'] & FT^2(A) \arrow[d, "F(e_A)"] \\
T'^2F(A) \arrow[r, "e'_{F(A)}"] & F(A)
\end{tikzcd}$$

$$\rightsquigarrow L^*_* \xrightarrow{L^*_*(F)} L^*_*(\mathbb{A}')$$

27. Categories over complexes

Let K be a locally finite simplicial complex, $\mathbb{A}$ be an additive category with chain duality. $\Rightarrow \mathbb{A}^*(K), \mathbb{A}_*(K)$.

2 purposes:

$\mathbb{A}^*(\Delta^k)$ multipl duality

$\mathbb{A}_*(K)$ local duality in K (or its failure)

K triangulated n-manifold

(1) $\forall \sigma \in K$ is $|\sigma|$-dimensional manifold with boundary

(2) $\forall \sigma n K D(\sigma, K)$ i $(n - |\sigma|)$-dimensional manifold with boundary

Definition 27.1. $M \in \mathbb{A}$ is <u>K-based</u> if

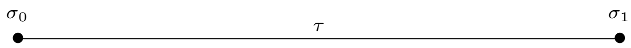
$$M = \sum_{\sigma \in K} M(\sigma) \qquad |\{\sigma \in K \mid M(\sigma) \neq 0\}| < \infty$$

A morphism of K-based objects.

$$f\colon M \to N, f = \{f(\tau, \sigma)\colon M(\sigma) \to N(\tau) \in \mathrm{Mor}(\mathbb{A}) \mid \sigma, \tau \in K\}$$

- $\mathbb{A}^*(K) =$ objects are K-bases objects in $\mathbb{A}$, morphisms are $\{f = \{f(\tau, \sigma) \mid f(\tau, \sigma) \neq 0 \Rightarrow \sigma \geq \tau\}\}$
- $\mathbb{A}_*(K) =$ objects are K-bases objects in $\mathbb{A}$, morphisms are $\{f = \{f(\tau, \sigma) \mid f(\tau, \sigma) \neq 0 \Rightarrow \sigma \leq \tau\}\}$

Example 27.2. Chain complexes in $\mathbb{B}(\mathbb{A}^*_*(K))$ $K = \Delta^1, C \in \mathbb{B}, \mathbb{A}^*(\Delta^1)$



$$\begin{tikzcd}
C_r & C(\sigma_0) = \Delta_*(\sigma_0, \partial\sigma_0) \arrow[d] & C(\tau) = \Delta_*(\tau, \partial\tau) \arrow[dl] \arrow[d] \arrow[dr] & C(\sigma_1) = \Delta_*(\sigma_1, \partial\sigma_1) \arrow[d] \\
C_{r-1} & C(\sigma_0) = \Delta_*(\sigma_0, \partial\sigma_0) & C(\tau) = \Delta_*(\tau, \partial\tau) & C(\sigma_1) = \Delta_*(\sigma_1, \partial\sigma_1)
\end{tikzcd}$$

Example 27.3. $\Delta(K)$ is in $\mathbb{B}(\mathbb{Z}^*(K))$
$\Delta(K)(\sigma) = \Delta(\sigma, \partial\sigma)$ $K = \Delta^1$

$$\begin{tikzcd}
0 \arrow[d] & \mathbb{Z} \arrow[dl] \arrow[d] \arrow[dr] & 0 \arrow[d] \\
\mathbb{Z} & 0 & \mathbb{Z}
\end{tikzcd}$$

$\Delta(K')$ is in $\mathbb{B}(\mathbb{Z}_*(K)), \Delta(K')(\sigma) = \Delta(D, \sigma, K), \partial D(\sigma, K))$

$$\begin{tikzcd}
\mathbb{Z} \arrow[d] \arrow[dr] & 0 \arrow[d] & \mathbb{Z} \oplus \mathbb{Z} \arrow[dl] \arrow[d] \arrow[dr] & 0 \arrow[d] & \mathbb{Z} \arrow[dl] \arrow[d] \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z}
\end{tikzcd}$$

Proposition 27.4 (Ranicki 90)**.**

(1) *$C \in \mathbb{B}(\mathbb{A}^*_*(K))$ is contractible $\iff$ $C(\sigma)$ is contractible for all $\sigma \in K$*
(2) *$f\colon C \to D$ in $\mathbb{B}(\mathbb{A}^*_*(K))$ is an equivalence $\iff$ $\forall \sigma \in K f(\sigma, \sigma)\colon C(\sigma) \to D(\sigma)$ is an equivalence*

28. Duality in $\mathbb{A}^*_*(K)$

Definition/Proposition 28.1.

$$T_K\colon \mathbb{A}^*_*(K) \longrightarrow \mathbb{B}(\mathbb{A}^*_*)(K)$$

$$\mathbb{A}^*_*(K) \xrightarrow{sum} \mathbb{A}^*_*[K] \xrightarrow{shift} \mathbb{B}(\mathbb{A}^*_*)(K) \xrightarrow{T} \mathbb{B}(\mathbb{A}^*_*(K))$$

$$M \longmapsto [M]\colon \sigma \mapsto \underbrace{\sum_{\substack{\overline{\sigma} \leq \sigma \\ \overline{\sigma} \geq \sigma}} M(\overline{\sigma})}_{[M][\sigma]} \longmapsto \underbrace{[M]^*[K]}_{[M]^*[K](\sigma) = S^{\overline{+}|\sigma|}[M][\sigma]} \longmapsto T_k(M)$$

$$T_K(M)_r(\sigma) = T(\sum_{\overline{\sigma}\leq\sigma} M(\overline{\sigma}))_{r-|\sigma|}$$
$$d_{T_K M} = \textit{ exercise}$$

Example 28.2.
(1) $\Delta(K) \in \mathbb{B}(\mathbb{Z}^*(K))$, $T_K(\Delta(K))(\sigma) = \Delta^{|\sigma|-*}(\sigma), \Delta(K)(\sigma) = \Delta(\sigma, \partial\sigma)$
(2) $\Delta(K') \in B(\mathbb{Z}_*(K)), T_K(\Delta(K'))(\sigma) = \Delta^{-|\sigma|-*}(D(\sigma, K))$

Talk 11.2

29. Functoriality

Recall.

$$: \mathbb{B}(\mathbb{A}_*(K)) \longrightarrow \mathbb{B}(\mathbb{A}^*(K)),$$
$$C \mapsto [C]_*[K](\sigma) = \sum_{\sigma\leq\overline{\sigma}} S^{|\sigma|}C(\overline{\sigma})$$

Observation

$$[C]_*[K] = \sum_{\overline{\sigma}\in K} (\Delta(\Delta^{|\sigma|}) \otimes C(\overline{\sigma}))$$

Proposition 29.1. *There exists* $\beta_C : [C]_*[K] \xrightarrow{\simeq} C, a \otimes b \mapsto \varepsilon(a)b$ *in* $\mathbb{B}(\mathbb{A})$

Proposition 29.2. *A simplicial map* $f : J \to K$ *induces a functor of additive categories with chain duality*

$$f^* : \mathbb{A}^*(K) \to \mathbb{A}^*(J)$$
$$f_* : \mathbb{A}_*(J) \to \mathbb{A}_*(K)$$

Proof. In the special case $K = *$ $f_* : A_*(J) \to \mathbb{A}$. Let $M \in A_*(J)$. $f_*(M) \in \mathbb{A}$

$$G(M) : Tf_*(M) \xrightarrow{\simeq} f_*T_J(M)$$

"via β"

$$f_*T_J(M) = f_*T([M]_*[K]) = T(f_*([M]_*[K]))$$

$$\beta_M : [M]_*[K] \xrightarrow{\simeq} M$$

$$T(\beta_M) : T(M) \xrightarrow{\simeq} T([M]_*[K])$$

∎

30. Symmetric and quadratic complexes over K

Recall.

$$\varphi : C^{\text{sing}}(X) \to \text{Hom}_{\mathbb{Z}_2}(W, C^{\text{sing}}(\widetilde{X}) \otimes_{\mathbb{Z}\pi} C^{\text{sing}}(\widetilde{X}))$$

$\varphi_0(C) = -capC$

Let K be a simplicial complex, $\Delta(K') \in \mathbb{B}(\mathbb{Z}_{*(K)})$ then $C = C^{\text{sing}}(K') \in \mathbb{B}(\mathbb{Z}_{*(K)})$ by $C(\sigma) = C^{\text{sing}}(D(\sigma, K), \partial D(\sigma, K))$

Want $\varphi^k : C \to \text{Hom}_{\mathbb{Z}_2}(W(\underbrace{C \otimes_{Z_*(K)} C}_{\text{Hom}_{Z_*(K)}(TC,C)})$

$$\begin{array}{ccc} \varphi_K(C)_0(\sigma) = (- \cap \partial_0 C) : \Sigma^k TC(\sigma) & \longrightarrow & C(\sigma) \\ \downarrow = & & \downarrow = \\ C^{n-|\sigma|}(D(\sigma)) & \longrightarrow & C(D(\sigma), \partial D(\sigma)) \end{array}$$

For all σ there exists $\partial_\sigma\colon C \to S^{|\sigma|}C(\sigma)$ chain map, $\sigma = \langle v_0, \dots v_{|\sigma|}\rangle, \sigma_i = \langle v_0, \dots v_i\rangle$

$$\partial_\sigma\colon C_n = \Sigma?? \xrightarrow{\text{proj}} C(\sigma_0)_n \xrightarrow{d_1} C\sigma_{1\,n-1} \xrightarrow{d_2} \dots \to C(\sigma_{|\sigma|})_{n-|\sigma|}$$

$\mathrm{Hom}_{\mathbb{Z}_2}(W, C \otimes_{Z_*(K)} C)$
$M \otimes_{Z_*(K)} N = \sum_{\sigma\in K}\sum_{\sigma\leq\lambda??\mu} S^{|\sigma|}(M(\lambda)\otimes_{\mathbb{Z}} N(\mu)) = \sum_{\sigma\in K} S^{|\sigma|}[M][\sigma]\otimes[N][\sigma]$
$C \otimes_A C$

$$C \simeq [C]_*[K] = \sum_{\sigma\in K} S^{|\sigma|}C(D(\sigma)) \xrightarrow{\Sigma S^{|\sigma|}\varphi??} \sum_{\sigma\in K} S^{|\sigma|}(C(D(\sigma)) \otimes C(D(\sigma)))$$

Proposition 30.1. *Let M n-manifold. $\nu\colon M \to |K|, r \pitchfork D(\sigma, K)$. Then let $C = \Sigma C(\sigma), C(\sigma) = C(M(\sigma), \partial M(\sigma))$ with $M(\sigma) = r^{-1}(D(\sigma, K))$*

$$\begin{array}{ccc} \mathrm{sign}_K^{\mathbb{L}^\bullet}(M) & \in & L^n(Z_*(K)) \\ \big\downarrow & & \big\downarrow A \\ \mathrm{sign}_{\mathbb{Z}\pi}^{L^\bullet}(M) & \in & L^n(\mathbb{Z}[\pi(K)]) \end{array}$$

Let $(f,b)\colon M \to X$ a degree one normal map, $r\colon X \to K, r \pitchfork D(\sigma, K), f\circ r \pitchfork D(\sigma, K)$. Then

$$\begin{array}{ccc} \mathrm{sign}_K^{\mathbb{L}^\bullet}(f,b) & \in & L^n(Z_*(K)) \\ \big\downarrow & & \big\downarrow A \\ \mathrm{sign}_{\mathbb{Z}\pi}^{L^\bullet}(f,b) & \in & L^n(\mathbb{Z}[\pi(K)]) \end{array}$$

Definition/Proposition 30.2. *For K $\pi_1(K) \xrightarrow{\alpha} \pi$*

$$A_\alpha\colon \mathbb{Z}_*(K) \longrightarrow \mathbb{Z}\pi,$$

$$M \mapsto \sum_{\widetilde{\sigma}\in\widetilde{K}} M(p\widetilde{\sigma})$$

Talk 12.1

31. Generalized (co-)homology theories

31.1. **Bordism spaces and spectra.** Cat = Diff, PL, TOP

Recall.

$$\Omega_n^{\mathrm{SCAT}}(K) = \{f\colon M \to K\}_{/\sim}$$

where $\sim$ means bordism $/K$.

Definition 31.1.
$(\mathbf{\Omega}_n^{\mathrm{SCAT}}(K))^{(k)} = \{f\colon (M, \partial_\sigma M) \to K \mid (M, \partial_\sigma M)(n+k)\text{-dim. SCAT mfd } (k+2)\text{-ad}\}$

$\emptyset$ SCAT n-manifold which is the base point for all n.

Proposition 31.2.
(1) $\Omega_{n+k}^{\mathrm{SCAT}}(K) \cong \pi_k(\mathbf{\Omega}_n^{\mathrm{SCAT}}(K))$
(2) $\mathbf{\Omega}_n^{\mathrm{SCAT}}(K) \simeq \mathbf{\Omega}_{n-1}^{\mathrm{SCAT}}(K)$

Definition 31.3. For $n \in \mathbb{Z}$ define

$$\mathbb{L}^n(\mathbb{A})^{(k)} = \{\text{ n-dim SAPC in } \mathbb{B}(\mathbb{A}^*(\Delta^k))\}$$
$$\mathbb{L}_n(\mathbb{A})^{(k)} = \{\text{ n-dim QAPC in } \mathbb{B}(\mathbb{A}^*(\Delta^k))\}$$

Proposition 31.4.

$$\pi_k(\mathbb{L}^n_n(\mathbb{A})) = L^{n+k}_{n+k}(\mathbb{A})$$
$$\mathbb{L}^n_n \simeq \Omega\mathbb{L}^{n-1}_{n-1}(\mathbb{A})$$

Definition/Proposition 31.5. *We have maps of spectra*

$$\mathrm{sign}^{\mathbb{L}^\bullet} : \mathbf{\Omega}^{\mathrm{STOP}}_\bullet \longrightarrow \mathbf{\Omega}^P_\bullet \longrightarrow \mathbb{L}^\bullet$$

$$M \longmapsto \underbrace{\mathrm{sign}^{\mathbb{L}^\bullet}(M)}_{sym.\ construction\ over\ Z^*(\Delta^k)}$$

of spaces

$$\begin{tikzcd} \mathrm{sign}^{\mathbb{L}^\bullet} : \mathrm{Sing}(\mathrm{G/TOP}) \arrow[r] \arrow[d, "\simeq"] & \mathbb{L}_0 \\ \widetilde{\mathcal{N}}^{\mathrm{TOP}}(*) & \end{tikzcd}$$

$$\alpha : \Delta^l \to GTOP \simeq (f(\alpha), b(\alpha)) : (M, \partial_i M) \to (\Delta^l, \partial_i \Delta^l)\ \textit{deg. 1. nor.} \longmapsto (C, \psi)$$

where $C(\sigma) = \mathscr{C}(f(\alpha)(\sigma)^!), \sigma < \Delta^l$ *quadratic construction over* $\mathbb{Z}^*(\Delta^k)$

Talk 12.2

32. Orientation intermezzo

Let X be a triangulated manifold.

$$\begin{tikzcd} \mathscr{S}^{\mathrm{TOP}}(X) \arrow[r] & N^{\mathrm{TOP}}(X) \arrow[r, "\mathrm{sign}^{L\bullet}_{\mathbb{Z}\pi}"] \arrow[d, "\cong"] & L_n(\mathbb{Z}\pi) \\ & {[X, \mathrm{G/TOP}]} \arrow[d, "\cong"] & \\ & {[X; \mathbb{L}_0\langle 1\rangle]} \arrow[d, "="] \arrow[r, phantom, "="] & \\ & \pi_0(\mathbb{L}_0\langle 1\rangle^X) \arrow[r, "\theta"] & \pi_k(\mathbb{L}_\bullet(\mathbb{Z}\pi)) \arrow[uuu] \end{tikzcd}$$

delooping ok, problem: $\mathcal{N}^{\mathrm{TOP}}(X)$ is contravariant, L_n is covariant

On the other hand

$$\mathrm{sign}^{L\bullet}_X : \underbrace{\mathcal{N}^{\mathrm{TOP}}(X)}_{\text{contravariant, local}} \to \underbrace{L_n(Z_*(X))}_{\text{local, covariant}}$$

Question. Isn't this a kind of Poincaré duality?

33. S-duality and Poincaré duality

Let $\mathbb{E}$ be an Ω-spectrum of Δ-sets. $\mathbb{E}_n \simeq \Omega\mathbb{E}_{n-1}$, K, L simplicial complexes.

$$H^n(K, \mathbb{E}) = [K_+, \mathbb{E}_-n]$$
$$H_n(K, \mathbb{E}) = \pi_n(K_+ \wedge \mathbb{E})$$

S-duality. For all K there exists a $K^a st$ and a map $\alpha\colon S^N \to K \wedge K^*$ such that

$$\alpha \setminus -\colon H^n$$

Slogan 1. H_* is H^*(S-dual).

There exists a combinatorial description of the S-duality $K \hookrightarrow \partial\Delta^{n+1}$

$$(\partial\Delta^{m+1})^{(m-l)}(\Sigma^m)^{(l)}$$
$$\sigma \leftrightarrow \sigma^*$$
$$\sigma \leq \tau \leftrightarrow \sigma^* \geq \tau^*$$

Definition 33.1 (Supplement)**.** Let $K' \subset (\Sigma^m)'$. Define the supplement $\overline{K} \subset (\Sigma^m)'$ by

$$\overline{K} = \{\tau \in (\Sigma^m)' \mid \text{ no face of } \tau \text{ is in } K\ \}$$

$\Sigma/\overline{K} = N/\partial N$ where N is regular neighbourhood of K.

Theorem 33.2. *$\Sigma^m/\overline{K}$ is S-dual of K*

Corollary 33.3.

$$H_n(K, \mathbb{E}) \cong H^{m-n}(\Sigma^M/\overline{K}; \mathbb{E})$$
$$\underbrace{\sigma^*}_{m-|\sigma|\text{-}dim.} \mapsto x(\sigma) \in (\mathbb{E}_{n-m})^{(m-|\sigma|)}$$

So for $\mathbb{E} = \mathbb{L}_\bullet$ or $= \mathbb{L}^\bullet$ the Δ-set $(\mathbb{L}_{n-m})^{(m-|\sigma|)}$ consists of geometric $n-|\sigma|$-dimensional $m-|\sigma|+2$-ads.

Proposition 33.4. (1) $\mathbb{L}_\bullet(\mathbb{A})^{K_+} \simeq \mathbb{L}_\bullet(\mathbb{A}^*(K))$
(2) $K_+ \wedge \mathbb{L}_\bullet(\mathbb{A}) \simeq \mathbb{L}_\bullet(\mathbb{A})^{\Sigma^m/\overline{K}} \simeq \mathbb{L}_\bullet(\mathbb{A}_*(K))$

Proof. (1) LHS $= \sigma \mapsto$ 0-dim. $(|\sigma|+2)$-ad QAPC in $\mathbb{A}^*(\Delta^{|\sigma|})$ = 0-dim.QAPC in $\mathbb{A}^*(K)$
(2)

$$\mathbb{A}_*(K) \cong \mathbb{A}^*(\Sigma^m/\overline{K})$$
$$\sigma \leftrightarrow \sigma^*$$
$$\sigma \leq \tau \leftrightarrow \sigma^* \geq \tau^*$$

□

34. Wrap up

Definition/Proposition 34.1. (1) *We have* $\mathrm{sign}_K^{\mathbb{L}^\bullet}\colon \Omega_n^{\mathrm{STOP}} \to L^n(\mathbb{Z}_*(K)) = H_n(K, \mathbb{L}^\bullet)$
(2) $\mathrm{sign}_K^{\mathbb{L}_\bullet}\colon \mathcal{N}^{\mathrm{TOP}}(K) \to L_n(\mathbb{Z}_*(K) \cong) H_n(K, \mathbb{L}_\bullet)$
(3) *(K triangulated mfd)*

$$\begin{array}{ccc}
\mathcal{N}^{\mathrm{TOP}}(K) & \xrightarrow{\mathrm{sign}_{\mathbb{Z}\pi}^{L_\bullet}} & L_n(\mathbb{Z}\pi) \\
{\scriptstyle \mathrm{sign}_K^{\mathbb{L}_\bullet}}\downarrow & & \downarrow{\scriptstyle =} \\
H_n(K, \mathbb{L}_\bullet) & \xrightarrow{A} & L_n(\mathbb{Z}\pi)
\end{array}$$

s

Talk 13

Recall. X manifold (triangulated or $r\colon X \to K$)

$$\begin{tikzcd}
 & \mathscr{S}^{\mathrm{TOP}}(X) \arrow[r] & \mathcal{N}^{\mathrm{TOP}}(X) \arrow[r, "\mathrm{sign}^{L_\bullet}_{\mathbb{Z}\pi}"] \arrow[d, "\mathrm{sign}^{L_\bullet}_X"] & L_n(\mathbb{Z}\pi) \arrow[d, "="] \\
\mathbb{S}_{n+1}(X) \arrow[r] & H_n(X, \mathbb{L}_\bullet) \arrow[r, "A"] & L_n(\mathbb{Z}\pi) \arrow[r] & \underbrace{\mathbb{S}_n(X)}_{???}
\end{tikzcd}$$

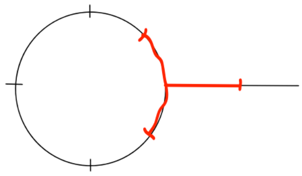
defined
for any
simplicial
complex X

Question. What local structure does an n-GPC X have?

Answer. It is locally a GNC = geometric normal complex

35. Geometric normal complexes

Remark 35.1. X n-GPC is not locally Poincaré



Definition 35.2. An n-GNC is a triple (Y, ν, ρ)
- Y locally finite simplicial complex
- $\nu\colon Y \to \mathrm{BSG}(k)$ $k-1$-dimensional spherical fibration
- $\rho\colon S^{n+k} \to \mathrm{Th}(\nu)$

Example 35.3. X n-GPC then (X, ν_X, ρ_X) is a GNC where ν_X is the SNF.

Definition 35.4. An $(n+1)$-GNP is $((Z(Y), \nu, \rho))$ such that
- ((Z,Y)) pair
- $\nu\colon Z \to \mathrm{BSG}(k)$, $\nu|_Y\colon Y \to \mathrm{BSG}(k)$
- $\rho\colon (D^{n+1+k}, S^{n+k}) \to (\mathrm{Th}(\nu), \mathrm{Th}(\nu|_Y))$

An $(n+1)$-geometric normal cobordism between n-dimensional GNC (X, ν, ρ) and (X', ν', ρ') is a $(n+1)$-GPC $((Z, X \amalg X'), \nu'', \rho'')$ such that $\nu''|_X = \nu, \nu''|_{X'} = \nu'$ and $\rho''|_{S^{n+k}_+} = \rho, \rho''|_{S^{n-k}_-} = \rho'$

Definition 35.5.

$$\Omega^N_n(K) = \{(X, \nu, \rho) n - GNC, \nu\colon X \to K\}_{/\text{ normal cob.}}$$

$$\begin{aligned}\mathbf{\Omega}^N_n(K)^{(k)} := \big\{&(X, \nu, \rho) \mid X \text{ simplicial } k+2\text{-ad }, \nu\colon X \to \mathrm{BSG}(r), \rho\colon \Delta^{n+k+r} \to \mathrm{Th}(\nu) \\ &\text{such that } \rho|_{\partial_i \Delta^{n+k+r}} \subseteq \mathrm{Th}(\nu|_{\partial^i X})\}\end{aligned}$$

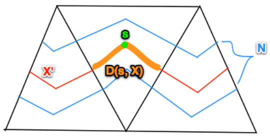
$$\mathbf{\Omega}^N_\bullet = \big\{\mathbf{\Omega}^N_{-n}\big\}_n \text{ is an } \Omega\text{-spectrum}$$

Definition/Proposition 35.6. *X n-GPC. There exists*

$$\mathrm{sign}^{\mathbf{\Omega}^N_\bullet}_X(X) \in H_n(X; \mathbf{\Omega}^N_\bullet) \ni (\sigma \mapsto (X(\sigma), \nu(\sigma), \rho(\sigma))) \in (\mathbf{\Omega}^N_{n-m})^{(m-|\sigma|)}$$

an $n - |\sigma|$-dimensional $m - |\sigma| + 2$-ad

Proof. Choose $X \subset \partial\Delta^{m+1}$. then $\rho_X\colon \Sigma^m \to \Sigma^m/\overline{X} \simeq \mathrm{Th}(\nu_X)$ $X(\sigma) = D(\sigma, X)$,



$\nu(\sigma) = \nu_X | X(\sigma), \rho(\sigma) = PT\colon \sigma^* = \Delta^{m-|\sigma|} \to \mathrm{Th}(\nu(\sigma))$ $N(\sigma) = \sigma^* \cap N$

$$\begin{tikzcd}
N(\sigma) \arrow[rr] \arrow[dr, dashed] \arrow[ddr] & & N \arrow[dr] & \\
& D(\nu(\sigma)) \arrow[rr] \arrow[d] & & D(\nu_X) \arrow[d] \\
& X(\sigma) \arrow[rr] & & X
\end{tikzcd}$$

□

36. Extracting algebra from a GNC

Recall. X n-GPC, $\nu(X), \rho_X\colon S^{n+k} \to \mathrm{Th}(\nu_X)$ $[X] = h(\rho_X) \cap u(\nu_X)$

$$\alpha_X\colon S^{n+k} \xrightarrow{\rho_X} \mathrm{Th}(\nu_X) \xrightarrow{\overline{\Delta}} X_+ \wedge \mathrm{Th}(\nu_X)$$

$$\begin{tikzcd}
C^{n-*}(X) \arrow[rr, "-\cup u(\nu_X)"] \arrow[dr, "-\cap[X]"] & & C^{n+k-*}(\mathrm{Th}(\nu_X)) \arrow[dl, "\alpha_X \backslash -"] \\
& C_*(X) &
\end{tikzcd}$$

$$[X] = h(\rho) \cap u(\nu) \rightsquigarrow - \cap [X]\colon \underbrace{C^{n-*} \to C(X)}_{\text{not } \simeq \text{ anymore}}$$

Slogan 2. n-GNC is a "want to be" n-GPC

$$\alpha_{\mathrm{Th}(\nu)}\colon S^N \longrightarrow \mathrm{Th}(\nu) \wedge \mathrm{Th}(\nu)^*$$

$$\neq$$

$$\overline{\Delta} \circ \rho\colon S^{n+k+p} \xrightarrow{\rho} \Sigma^p\, \mathrm{Th}(\nu) \xrightarrow{\overline{\Delta}} \Sigma^p X_+ \wedge \mathrm{Th}(\nu)$$

$$\alpha_{\mathrm{Th}(\nu)} \backslash -\colon [\mathrm{Th}(\nu)^*, \Sigma^p X_+] \longrightarrow [S^n, \Sigma^p X_+ \wedge \mathrm{Th}(\nu)],$$
$$\Gamma_X \xrightarrow{\simeq} \overline{\Delta} \circ \rho$$

$$\begin{tikzcd}
C^{n-*}(X) \arrow[r, "-\cup u(\nu_X)", "\simeq"'] \arrow[d, "-\cap[X]"] & C^{n+k-*}(\mathrm{Th}(\nu_X)) \arrow[r, "\alpha_{\mathrm{Th}(\nu)}", "\simeq"'] & C_{*+p}(\mathrm{Th}(\nu)^*) \arrow[d, "(\Gamma_X)_*"] \\
C(X) \arrow[rr, "\Sigma^p"] & & C_{*+p}(\Sigma^p X_+)
\end{tikzcd}$$

Structures

$$\begin{tikzcd}
\overset{u(\nu)}{H^k(\mathrm{Th}(\nu))} \arrow[rr, "\alpha_{\mathrm{Th}(X)}-"] \arrow[d, "-\cap h(\rho)"] & & H_{n+p}(\mathrm{Th}(\nu)^*) \arrow[dl, "(\Gamma_X)_*"] \arrow[d, "\varphi"] \\
H_n(X) \arrow[r] \arrow[d, "\varphi"] & H_{n-|\sigma|}(\Sigma^p X) \arrow[d] & Q^{n+p}(C(\mathrm{Th}(\nu)^*)) \arrow[dl, "\Gamma_X\otimes\Gamma_X"] \arrow[d] \\
Q^n(C(X)) \arrow[r] \arrow[d, "J"] & Q^{n-p}(C(\Sigma^p X)) \arrow[d] & \widehat{Q}^{n+p}(C(\mathrm{Th}(\nu)^*)) \arrow[dl, "\widehat{\Gamma_X\otimes\Gamma_X}"] \arrow[d, "\cong"] \\
\widehat{Q}^n(C(X)) \arrow[r] & \widehat{Q}^{n+p}(C(\Sigma^p X)) & \widehat{Q}^{n+p}(C^{N-*}(\mathrm{Th}(\nu))) \arrow[d, "\cong"] \\
 & \widehat{Q}^n(C^{n-x}(X)) \arrow[ul, "(\varphi[X])_0"] \arrow[r] & \widehat{Q}^{n+p}(C^{n+p-*}(X)) \arrow[ul, "S^p(\varphi[X]_0)"] \\
 & & \widehat{Q}^0(C^{-*}(X)) \arrow[u, "\cong"] \arrow[ul, "S^n"]
\end{tikzcd}$$

$$(\varphi[X])_0 = -\cap[X]\colon C^{n-*}(X) \to C(X)$$
$$S^p\varphi([X])_0 = C^{n+p-*}(X) \to \Sigma^p C(X)$$

Definition 36.1. Let $C \in \mathbb{B}(\mathbb{A})$. A chain bundle on C is a cycle $\gamma \in \mathrm{Hom}(\widehat{W}, C^{-*}\otimes_{\mathbb{A}} C^{-*})_0$

Definition 36.2. An n-dim ANC is a 4-tuple $(C, \varphi, \gamma, \chi)$ such that

- C in $\mathbb{B}(\mathbb{A})$
- φ n-cycle in $\mathrm{Hom}(W, C\otimes_A C)$
- γ 0-cycle in $\mathrm{Hom}(\widehat{W}, C^{-*}\otimes_{\mathbb{A}} C^{-*})$
- χ(n+1)-chain in $\mathrm{Hom}(\widehat{W}, C\otimes_{\mathbb{A}} C)$

such that $\widehat{\varphi}_0(S\gamma) - J(\varphi) = d\chi$ where $\widehat{\varphi}_0 = \widehat{\varphi_0\otimes\varphi_0}$

Definition/Proposition 36.3. *X n-GPC we obtain*

$$\mathrm{sign}_X^{\mathbb{NL}}(X) \in H_n(X; \mathbb{NL})$$

$$\mathbb{NL}^n(\mathbb{A}) = \text{ cobordism groups of ANCs}$$
$$\Omega_n^N \to \mathbb{NL}^n(\mathbb{Z})$$
$$\exists\,\mathrm{sign}^{\mathbb{NL}}\colon \mathbf{\Omega}^N \to \mathbb{NL}$$

Talk 14

Let X be a finite simplicial complex,
$\mathbf{\Omega}^{P_n}(X) =$ bordism of n-dimensional n-GPC Q with map $g\colon Q \to X$.
$\mathbf{\Omega}^{N_n}(X) =$ bordism of n-dimensional GNC Q with map $g\colon Q \to X \cong H_n(X, \mathbf{\Omega}^N(\mathrm{pt})) \cong H_n(X, \mathrm{MSG})$

$$
\begin{tikzcd}
\mathbb{S}_{n+1}(X) \arrow[dr] \arrow[rr, bend left] & & H_n(X;\boldsymbol{\Omega}^P) \arrow[dr] \arrow[rr, bend left] & & L_{n-1}\mathbb{Z}\pi \\
& H_n(X,\mathbb{L}_\bullet) \arrow[ur] \arrow[dr] & & \Omega_n^P(X) \arrow[ur] \arrow[dr] & \\
\Omega_{n+1}^N(X) \arrow[ur] \arrow[rr, bend right] & & L_n\mathbb{Z}\pi \arrow[ur] \arrow[rr, bend right] & & \mathbb{S}_n(X)
\end{tikzcd}
$$

$$\Omega_{n+1}^N(X) \to L_n(\mathbb{Z}\pi)$$

$$(Q^{n+1}(g))GNC \mapsto ANC \mapsto \partial(C(Q),\varphi_Q,\gamma_Q,X_Q) = (D,\psi)AQPC$$

where $D = \mathscr{C}(\varphi_Q)_0\colon C(Q)^{n+1-*} \to C(Q)_{*+1}$

$$
\begin{array}{c}
H_n(X,\boldsymbol{\Omega}^p) = \left\{ \begin{array}{c} \text{bordism of } Q\text{- } n\text{-GPC,} \\ g\colon Q \ \to \ X \text{ such} \\ \text{that for each simplex} \\ \sigma \in X(g^{-1}D(\sigma), g^{-1}\partial D(\sigma)) \\ \text{is an } n-|\sigma|\text{-GPP} \end{array} \right\} \\
\Big\downarrow{\scriptstyle A} \\
\Omega_n^P(X) = \{Qn - GPC, g\colon Q \to X\}
\end{array}
$$

Example 36.4. If X is an n-GPC $(X, 1\colon X \to X) \in \Omega_n^p(X)$, $s(X, !\colon X \to X) = s(x) \in \mathbb{S}_n(X)$

Orientations for generalized homology theories

$$(M,1) \in \Omega_n^{\mathrm{SO}}(M) \cong H_n(M;\boldsymbol{\Omega}^{\mathrm{SO}}(\mathrm{pt}))$$

h_* a generalized homology theory

$$[M] \in h_n(M) \to H_n(M, M - \{x\}))$$

$$
\begin{tikzcd}
& H_n(X;\mathbb{L}^\bullet(\mathbb{Z})) \arrow[dr] \arrow[rr, bend left] & & H_n(X;\mathbb{NL}^\bullet(\mathbb{Z})) \arrow[dr] & \\
H_n(X;\mathbb{L}_\bullet(\mathbb{Z})) \arrow[ur] \arrow[dr] & & VL^n(\mathbb{Z},X) \arrow[ur] \arrow[dr] & & H_{n-1}(X;\mathbb{L}_\bullet(\mathbb{Z})) \\
& L_n\mathbb{Z}\pi \arrow[ur] \arrow[rr, bend right] & & \mathbb{S}_n(X) \arrow[ur] &
\end{tikzcd}
$$

$$C = \text{f.g. } \mathbb{Z}\pi\text{-module chain complex}$$
$$\varphi \in Q^n(C)$$

$$\begin{aligned} C \otimes_{\mathbb{Z}\pi} C &= \mathbb{Z}[\mathbb{Z}_2]\text{-module chain complex} \\ &= C \otimes_{\mathbb{Z}} C / \{gx \otimes y - x \otimes gy \mid x \in C, y \in Cg, \in \pi\} \\ &= C \otimes_{\mathbb{Z}} C / \left\{x \otimes y - g^{-1}x \otimes gy \mid x \in C, y \in Cg, \in \pi\right\} \end{aligned}$$

$T(x \otimes y) = \pm y \otimes x$

$$\widetilde{\Delta}_0 \colon C(\widetilde{X}) \to C(\widetilde{X}) \otimes C(\widetilde{X})$$

induced by $\Delta \colon X \to \widetilde{X} \times \widetilde{X}, x \mapsto (x, x)$ [...]

Let

$$P \colon \ldots \to P_2 \to P_1 \to P_0$$

be a free $\mathbb{Z}\pi$-module resolution of $\mathbb{Z}$ with trivial π-action

[...]

$$\begin{tikzcd} H_n(X) \arrow[r, "V\varphi"] \arrow[dr, "\varphi"] & \mathrm{VQ}^n(C(\widetilde{X})) \\ & Q^n C(\widetilde{X}) \end{tikzcd}$$

A symmetric from (a_{ij}) over $\mathbb{Z}\pi$ has $a_{ij} = \overline{a}_{ij}$

Definition 36.5. The form is visible if $a_{ij} \in \mathbb{Z} \subset \mathbb{Z}\pi / \{x + \overline{x}\}$

Example 36.6. $\pi = \mathbb{Z}_2$, $\mathbb{Z}\pi = \mathbb{Z}[\mathbb{Z}_2] = \mathbb{Z}[t]/(t^2 - 1)$ The symmetric form $(\mathbb{Z}\pi, t = \overline{t} = t^{-1})$ is not visible

$$\Omega^P_{4k}(B\mathbb{Z}) \xrightarrow{\text{sign}^{\mathbb{L}^\bullet}} L^{4k}(\mathbb{Z}[\mathbb{Z}_2]) \ni (C, t)$$

is not in the image

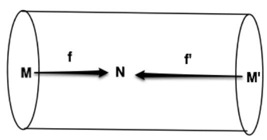
If $X = B\pi$

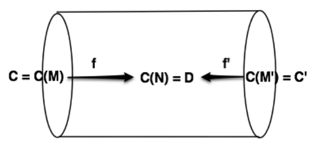
$$Q^n_{(\mathbb{Z}, B\pi)}(C) = \mathrm{VQ}^n(\text{ assembly of C })$$

$$\mathrm{VL}^n(\mathbb{Z}, X) = \left\{ \begin{array}{c} \text{cobordism of C = chain complex on} \\ Z^*(X) \text{ with visible symmetric Poincaré} \\ \text{structure } \varphi \in Q^n_{(\mathbb{Z},X)})(C) \text{ such that} \\ \varphi_0 \colon C(\widetilde{X})^{n-*} \xrightarrow{\simeq} C(\widetilde{X})\mathbb{Z}[\pi_1(X)]\text{-module} \\ \text{chain equivalence (ie globally Poincaré)} \end{array} \right\}$$

$$\varphi \in Q^n(C) = H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_{Z_* X} C))$$

Talk 15





Definition 36.7. An $(n+1)$-dimensional SAP cobordism

$$(\delta\varphi, \varphi \oplus -\varphi) \in Q^{n+1}((f, f')\colon C \otimes C' \to D)$$

where the chain map $\begin{pmatrix} \delta\varphi_0 \\ \varphi_0 f \end{pmatrix} \colon \mathscr{C}(f')^{n-1-*} \to \mathscr{C}(f)$ is a chain equivalence

Theorem 36.8. *Can recover the $n+1$-dimensional SAP cobordism*

$$((f, f')\colon C \otimes C' \to D, \delta\varphi, \varphi \oplus -\varphi')$$

from the $(n+1)$-dimensional symmetric pair $(g\colon C \to \mathscr{C}(f') = D/C', \delta\varphi/\varphi', \varphi)$ (without Poincaré duality)

A chain map $g\colon C \to E(=\mathscr{C}(f'))$ induces a $\mathbb{Z}[\mathbb{Z}_2]$-module chain map

$$g \otimes g\colon C \otimes C \to E \otimes E$$

and hence $g^{\%}\colon Q^n(C) \to Q^n(E)$

$$\underbrace{Q^{n+1}(g)}_{H_{n+1}(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes C))} \to Q^n(C) \xrightarrow{g^{\%}} Q^n(E)$$

Theorem 36.9. *There is a 1-1 correspondence between homotopy equivalence classes of $(n+1)$-dimensional symmetric Poincaré cobordism*

$$((f, f')\colon C \otimes C' \to D), \delta\varphi, \varphi \oplus -\varphi') \in Q^{n+1}(f, f')$$

and $(n+1)$-dimensional symmetric pair $(g\colon C \to E, \delta\varphi', \varphi)$ (input for algebraic surgery).

$\Rightarrow$ $((f, f')\colon C \otimes C' \to D) \rightsquigarrow (g\colon C \xrightarrow{f} D \hookrightarrow E = \mathscr{C}(f'\colon C' \to D))$ with (C, φ) Poincaré .

$\Leftarrow$ Given symmetric pair $(g\colon C \to E, \delta\varphi', \varphi)$ with (C, φ) Poincaré define symmetric Poincaré cobordism $((f, f')\colon C \oplus C' \to D, \delta\varphi, \varphi \oplus -\varphi')$ by

[...]

$C' = \mathscr{C}(\begin{pmatrix} \delta\varphi_0 \\ \varphi_0 f \end{pmatrix} \colon E^{n+1-*} \to D)$

Comment: The want-to-be chain duality of the pair. So C' measures the failure of the pair to be Poincaré

Special case $C = 0$

$(n+1)$-dimensional sym. Poincaré pairs $(f\colon C' \to D, \delta, \varphi') \leftrightarrow (n+1)$-dimensional symmetric complexes $(E = \mathscr{C}(f'), \delta\varphi/\varphi')$

Everything works equally well in the quadratic category.

... induces morphisms $f\colon L_n(A) \to L_n(B); (C, \varphi) \mapsto B \otimes_A (C, \varphi)$ using the functor $B \otimes_A -\colon$ A-modules $\to$ B-modules $M \mapsto B \otimes_A M$ using $((B, A)$-bimodule) $B \times BA \to B; (b, x, a) \mapsto bxf(a)$

Want $L_n(f)$ to fit into

$$L_n(A) \to L_n(B) \to L_n(f) \to L_{n-1}(A) \to$$

with $L_n(f)$ relative cobordism group of pairs

$((C, \varphi) =$ $(n-1)$-dimensional quadratic Poincaré complex / A , $g\colon B \otimes_A C \to D, \delta\varphi, 1 \otimes \varphi)) =$ n-dimensional

$$L_n(B) \to L_n(f)\colon (D, \delta\varphi) \to (0, (0, 0) \to D, \delta\varphi, 0)$$
$$L_n(f) \to L_{n-1}(A)\colon (C, \varphi), (g\colon B \otimes AC \to D, \delta\varphi, 1 \otimes \varphi)) \mapsto (C, \varphi)$$

Definition 36.10. The boundary of an $(n+1)$-dimensional symmetric complex (E, θ) is the n-dimensional symmetric Poincaré complex $(\partial E, \partial\theta)$ with [...]

Consequence: an n-dimensional symmetric Poincaré complex (C,φ) over A is such that $(C,\varphi)=0\in L^*(A)$ if and only if (C,φ) is homotopy equivalent to the boundary $\partial(E,\theta)$ of an $(n+1)$-dimensional symmetric complex (E,θ).

Suppose now that $f\colon A\to B=S^{-1}A$ is the inclusion with $B=S^{-1}A$ the localization of A inverting the subset $S\subset A$ of central non-zero divisors $s\in A$

S multiplicative closed, $1\in S$

$B=S^{-1}A=\left\{a/s \mid \frac{a}{s}=\frac{at}{st} \text{ for } t\in A\right\}$ (fractions)

$L^n(A)\to L^n(S^{-1}A)\to L^n(A,S)=L^n(A\to S^{-1}A)\to L^{n-1}(A)$

Clearing fractions gives

$$\begin{array}{ccc}
L^n(S^{-1}A)=\left\{\begin{array}{c}\text{cobordism group of n-dimensional}\\ \text{symmetric } S^{-1}A\text{- Poincaré}\\ \text{complexes } (C,\varphi) \text{ over A}\end{array}\right\} & & S^{-1}A\otimes_A(C,\varphi)\\
\Big\downarrow \partial & & \Big\downarrow \\
L^n(A(S))\cong\left\{\begin{array}{c}\text{coboridsm group of}\\ (n-1)\text{-dimensional symmetric Poincaré}\\ S^{-1}A\text{-contractible complexes over A}\end{array}\right\} & & \partial(C,\varphi)
\end{array}$$

[...]

Talk 16.1

37. Quadratic and symmetric versus normal complexes

- use ∂ to explain the relation between L_n, L^n, NL^n ($\mathbb{A}$ - add. category with chain duality)

$C\in\mathbb{B}(\mathbb{A})$

$$\to Q_n(C)\xrightarrow{1+T}Q^n(C)\xrightarrow{J}\widehat{Q}^n(C)\xrightarrow{H}Q_{n-1}(C)$$

Last lecture

$$L_n(\mathbb{A})\xrightarrow{1+T}L^n(\mathbb{A})\to(\text{ relative term })\to L_{n-1}(\mathbb{A})$$

$(C\xrightarrow{f}D,\delta\varphi,(1+T)\psi)$ n-sym Poincaré pair, (C,ψ) $(n-1)$ quadratic Poincaré

Boundary of (C,φ) n-dimensional symmetric complex (non-Poincaré), $\varphi_0\colon C^{n-*}\to C_*$ is $(\partial C,\partial\varphi)$ where $\partial C=\Sigma^{-1}\mathscr{C}(\varphi_0)$ and

$$\partial C\to C^{n-k}\xrightarrow{\varphi}C\xrightarrow{e}\mathscr{C}(\varphi_0)$$

$$\begin{aligned}Q^n(C)&\xrightarrow{e^\%}Q^n(\mathscr{C}(\varphi_0))\\ \varphi&\mapsto e^\%(\varphi)\end{aligned}$$

$$\begin{aligned}Q^{n-1}(\partial C)&\xrightarrow{S}Q^n(S\partial C)\simeq Q^n(\mathscr{C}(\varphi_0))\\ \partial\varphi&\mapsto S(\partial\varphi)=e^\%(\varphi)\end{aligned}$$

Recall. n-ANC (C,φ,γ,χ)

- $C\in(\mathbb{B},\mathbb{A})$
- cycle $\varphi\in\operatorname{Hom}(W,C\otimes_{\mathbb{A}}C)$

...

There exists a map

$$L_n(\mathbb{A}) \to L^n(\mathbb{A}) \xrightarrow{J} NL^n(\mathbb{A}) \xrightarrow{J} L_{n-1}(\mathbb{A})$$

Proposition 37.1. *(C,φ) n-SAC in $\mathbb{A}$ (not Poincaré)*

(1) *there exists $\gamma \in \widehat{Q}^0(C^{-*})$ such that $\widehat{\varphi}_0^{\%}(S^n(\gamma)) = J(\varphi) \iff \exists \partial\psi \in Q_{n-1}(C)$ such that $(1+T)\partial\psi = \partial\varphi$*

Proof.

$$\begin{tikzcd}
 & & Q_{n-1}(\partial C) \arrow[r] & Q^{n-1}(\partial C) \arrow[r] & \widehat{Q}^{n-s}(\partial C) \arrow[d, "S"] \\
 & & & Q^n(C) \arrow[r, "e^{\%}"] & Q^n(\mathscr{C}(\varphi_0)) \arrow[d, "J"] \\
\widehat{Q}^0, C^{-*} \arrow[r] & \widehat{Q}^n(C^{n-*}) \arrow[rr, "\widehat{\varphi}_0^{\%}"] & & \widehat{Q}^n(C) \arrow[r, "\widehat{e}^{\%}"] & \widehat{Q}^n(\mathscr{C}(\varphi_0))
\end{tikzcd}$$

□

Talk 16.2

The formula for the quadratic boundary of an n-dimensional normal complex

$$(C,\varphi,\gamma,\chi)$$

$\chi_s : C^{n-r+s} \to C_{r+s}$ clutching function

We get a quadratic complex out of a normal complex in the following way

$$\partial(C,\varphi,\gamma,\chi) = (\mathscr{C}(\varphi_0 : C^{n-*} \to C)_{*+1}, \psi)$$

$$\psi_0 = \begin{pmatrix} \chi & ? \\ ? & ? \end{pmatrix} \qquad \text{[blue book for details]}$$

How does a n-dimensional GNC determine a n-dimensional ANC in $\mathbb{Z}_*(X)$?

We get φ by symmetric construction and the γ by the hyperquadratic construction

Interesting thing is how to get $\chi(\rho)$: J.W.C. Whitehead, annals paper "a certain exact sequence". For an arbitrary space T analyzed fibre of the Hurewicz map

$$\overline{H_n}(T) \to H_n(T)$$

$$\to \underbrace{\Gamma_n(T)}_{\text{quadratic structures}} \to \pi_n(T) \xrightarrow{h} H_n(T) \to \Gamma_{n-1}(T) \to$$

For any $\nu : X \to \mathrm{BSG}(k)$

$$(*) \qquad \varphi_0\gamma_{s-n}\varphi_0^* - \varphi_s = d\chi_s + \chi_s d' + \chi_{s-1} + \chi_{s-1}^* ??$$

$$\begin{tikzcd}
\Gamma_{n+k}T(\nu) \arrow[r] & \underbrace{\pi_{n+k}(T(\nu))}_{\ni\rho} \arrow[r, "h"] & H_{n+k}(T(\nu)) \cong H_n(X) \arrow[r] & \Gamma_{n+k-1}(T(\nu)) \arrow[d, "\theta_{T(\nu)}"] \\
 & \underbrace{Q_n(C(X),\gamma(\nu))}_{\ni(\varphi_X([X]),\chi(\rho)) \text{ abelian group of all } \varphi,\chi \text{ satisfying } (*)} \arrow[r] & Q^n(C(X)) \arrow[r, "J_{\gamma(\nu)}"] & \widehat{Q}^n(C(X))
\end{tikzcd}$$

$$\begin{tikzcd}
\Gamma_{n+k}T(\nu) \arrow[r] & \underbrace{\pi_{n+k}(T(\nu))}_{\ni\rho} \arrow[r, "h"] & H_{n+k}(T(\nu)) \cong H_n(X) \arrow[r] & \Gamma_{n+k-1}(T(\nu)) \arrow[d, "\theta_{T(\nu)}"] \\
& \underbrace{Q_n^{\mathbb{Z}_*X}(C(X),\gamma(\nu))}_{\ni(\varphi_X([X]),\chi(\rho))} \arrow[r] & Q^n_{\mathbb{Z}_*X}(C(X)) \arrow[r, "J_{\gamma(\nu)}"] & \widehat{Q}^n_{\mathbb{Z}_*X}(C(X))
\end{tikzcd}$$

For an n-dimensional GNC (X,ν,ρ) the image $(\theta,\phi,\rho) \in Q_n^{\mathbb{Z}_*X}(C(X),\gamma(\nu))$ defines an n-dimensional ANC in $\mathbb{Z}_*X$

$$(C(X),\varphi(X),\gamma(X),\chi(\rho))$$

with boundary $\partial(C,\varphi,\gamma,\chi) = (\partial C,\psi)$ an $(n-1)$-dimensional QAPC in $\mathbb{Z}_*X$ with assembly $A(\partial C) = \mathscr{C}([X]\cap : C(\widetilde{X})^{n-*} \to C(\widetilde{X}))$ in $\mathbb{Z}\pi X$

[...]

$$VL^n(\mathbb{Z}_*X) \to \mathbb{S}_n(X)$$

[...]