1. Manifold Atlas : Regensburg Surgery Blockseminar 2012
The s-cobordism theorem III (Francesca Diana and Matthias Blank)

In this talk we will give the definition of Whitehead torsion for chain complexes, for CW complexes and for h-cobordisms; after proving some of the main properties of this algebraic invariants, we will finally be able to prove the s-cobordism theorem.

Whitehead group. Let R be a ring with unit.

Definition 1.1. $\mathrm{GL}(R) \coloneqq \mathrm{colim}_n \mathrm{GL}(n, R)$

$$i\colon \mathrm{GL}(n,R) \longrightarrow \mathrm{GL}(n+1,R),$$
$$A \mapsto \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right)$$

$$K_1(R) \coloneqq \mathrm{GL}(R)/[\mathrm{GL}(R), \mathrm{GL}(R)]$$
$$\widetilde{K}_1(R) \coloneqq K_1(R)/\{[-1]\}$$

Definition 1.2 (Whitehead group)**.** Let G be a group. We define the **Whitehead group** of G ($\mathrm{Wh}(G)$) as the cokernel of the map:

$$G \times \{\pm 1\} \to K_1(\mathbb{Z}G)$$
$$(g, \pm 1) \mapsto \pm g$$

In other words, $\mathrm{Wh}(G) \coloneqq K_1(\mathbb{Z}G)/\{[\pm g] : g \in G\}$, where $\mathbb{Z}G$ denotes the group ring associated to the group G.

Lemma 1.3. *We have $E(R) = [\mathrm{GL}(R), \mathrm{GL}(R)]$, where $E(R) \subset \mathrm{GL}(R)$ subgroup generated by the elementary matrices, i.e. matrices of the form $I + rE_{ij} \quad (r \in R)$*

Proof. Exercise. □

Here are some examples of Whitehead groups:

Example 1.4. If $G = \mathbb{Z}^n$, then $\mathrm{Wh}(G)$ is trivial.

Example 1.5. $\mathrm{Wh}(\mathbb{Z}/p)$ is the free abelian group of rank $\frac{p-3}{2}$ for p odd prime, while $\mathrm{Wh}(\mathbb{Z}/2)$ is trivial.

Example 1.6 (Conjecture). The Whitehead group of a torsion free group is trivial.

Remark 1.7. This definition of $\mathrm{Wh}(G)$ is equivalent to the one given in the previous talk.

Whitehead torsion for chain complexes. Let $f_*\colon C_* \to D_*$ chain map between R-chain complexes.

Definition 1.8 (Mapping Cone)**.** We define $\mathrm{cone}_*(f_*)$ to be the R-chain complex with p-differential:

$$C_{p-1} \oplus D_p \xrightarrow{\begin{pmatrix} -c_{p-1} & 0 \\ f_{p-1} & d_p \end{pmatrix}} C_{p-2} \oplus D_{p-1}$$

Let (C_*, c_*) be a contractible, based free, finite (i.e. there exists a number N such that $C_p = 0, |p| > \mathrm{N}$ and every module C_p is finitely generated) R-chain complex. We consider:

$$(c_* + \gamma_*)_{\mathrm{odd}}\colon \bigoplus_{p\in\mathbb{Z}} C_{2p+1} =: C_{\mathrm{odd}} \to \bigoplus_{p\in\mathbb{Z}} C_{2p} =: C_{\mathrm{ev}}$$

We define A to be the matrix of $(c_* + \gamma_*)_{\text{odd}}$. One can show that A is invertible (see Remark 1.9), so that we have a well-defined class $[A] \in \widetilde{K}_1(R)$.

Remark 1.9. To show that A is invertible one has to multiply it by suitable triangular matrices having ones on the diagonal. With the same procedure, one can show that the definition of $[A]$ does not depend on the contraction chosen; in particular, if we take another chain contraction δ, the map $(c_* + \delta_*)_{\text{odd}}$ will give an invertible matrix which belong to the same class of A in $\widetilde{K}_1(R)$ (See pag. 28 of the notes by W.Lück for details).

Definition 1.10 (Whitehead torsion)**.** The **Whitehead torsion** can be defined as:

$$\tau(C_*) \coloneqq [A] \in \widetilde{K}_1(R)$$

If $f_* \colon C_* \to D_*$ is an homotopy equivalence between based free, finite R-chain complexes, then

$$\tau(f_*) \coloneqq \tau(\text{cone}_*(f_*)) \in \widetilde{K}_1(R)$$

Notice that, since f_* is an homotopy equivalence, $\text{cone}_*(f_*)$ is contractible.

Lemma 1.11. *Assume all the R-chain complexes are finite and based free. Then*

(1)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_* & \longrightarrow & D_* & \longrightarrow & E_* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle h_*} & & \\ 0 & \longrightarrow & C'_* & \longrightarrow & D'_* & \longrightarrow & E'_* & \longrightarrow & 0 \end{array}$$

If

- *the rows are based exact*
- *two maps between $f_*, g_* h_*$ are homotopy equivalences*

Then

- *the third map is also an homotopy equivalence*
- $\tau(g_*) = \tau(f_*) + \tau(h_*)$

(2) $f_* \simeq g_* \colon C_* \to D_*$ *homotopic R-chain homotopy equivalences. Then*

$$\tau(f_*) = \tau(g_*)$$

(3) $f_* \colon C_* \to D_*, g_* \colon D_* \to E_*$ *R-chain homotopy equivalences. Then:*

$$\tau(g_* \circ f_*) = \tau(g_*) + \tau(f_*)$$

Proof. See Lemma 2.9 in the notes by W.Lück. □

1.1. **Whitehead torsion for CW complexes.** Let X, Y be connected, finite CW complexes with universal coverings $p_X \colon \widetilde{X} \to X$ and $p_Y \colon \widetilde{Y} \to Y$ respectively. Let $f \colon X \to Y$ be an homotopy equivalence that maps the point $x = p_X(\widetilde{x})$ to the point $y = p_Y(\widetilde{y})$ for some fixed $\widetilde{x} \in \widetilde{X}$ and $\widetilde{y} \in \widetilde{Y}$. We consider the following diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle p_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

where $\widetilde{f}$ is the lift of f on the universal coverings such that $\widetilde{f}(\widetilde{x}) = \widetilde{y}$.
We have a $\mathbb{Z}[\pi_1(Y, y)]$-homotopy equivalence:

$$C_*(\widetilde{f})\colon C_*(\widetilde{X}) \to C_*(\widetilde{Y})$$

Notice that $\pi_1(X, x)$ is identified with $\pi_1(Y, y)$ by $\pi_1(f, x)$. Moreover $C_*(\widetilde{X})$ (resp. $C_*(\widetilde{Y})$) is a free chain complexes with base given by choosing a lift $\widetilde{e} \subset \widetilde{X}$ (resp. $\widetilde{e} \subset \widetilde{Y}$) for each cell $e \subset X$ (resp. $e \subset Y$) and choosing an orientation of e.

Then we define:

Definition 1.12 (Whitehead torsion)**.** The **Whitehead torsion** of f is:

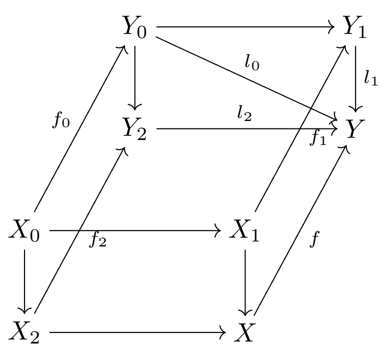
$$\tau(f) \coloneqq \tau(C_*(\widetilde{f})) \in \mathrm{Wh}(\pi_1(Y, y))$$

Remark 1.13. The definition of τ is independent of the choice of:

- lifts and orientation; indeed we are taking $\tau(f)$ in $\mathrm{Wh}(\pi_1(Y, y)) = K_1(\mathbb{Z}\pi_1(Y, y))/\{[\pm g] : g \in \pi_1(Y, y)\}$ and changing lifts or orientation just means that we are replacing $\widetilde{e}$ by $\pm g\widetilde{e}$ for some $g \in \pi_1(Y, y)$.
- the base point y; one can show that, for any two base points $y, y' \in Y$ there exists a unique isomorphism between $\mathrm{Wh}(\pi_1(Y, y))$ and $\mathrm{Wh}(\pi_1(Y, y'))$.

Theorem 1.14. *Assume all the CW complexes finite.*

(1) *Let the following be cellular pushouts of CW complexes:*



where f_0, f_1, f_2 are homotopy equivalences, and f is the map induced by f_0, f_1, f_2 and the pushout. Then
- *f is an homotopy equivalence*
- $\tau(f) = (l_1)_*\tau(f_1) + (l_2)_*\tau(f_1) - (l_0)_*\tau(f_0)$

where $(l_i)_* : \mathrm{Wh}(\pi_1(Y_i)) \to \mathrm{Wh}(\pi_1(Y))$ is the map induced by l_i on the Whitehead groups for $i = 0, 1, 2$.

(2) *Let $f \simeq g : X \to Y$ homotopic maps of CW complexes. Then:*
- $f_* = g_*\colon \mathrm{Wh}(\pi_1(X)) \to \mathrm{Wh}(\pi_1(Y))$
- *If f, g homotopy equivalences, $\Rightarrow \tau(f_*) = \tau(g_*)$.*

(3) *$f\colon X \to Y, g\colon Y \to Z$ homotopy equivalences $\Rightarrow \tau(g \circ f) = g_*\tau(f) + \tau(g)$*

(4) *$f\colon X' \to X, g\colon Y' \to Y$ homotopy equivalences of connected CW complexes. Then:*

$$\tau(f \times g) = \chi(X)j_*\tau(g) + \chi(Y)i_*\tau(f)$$

where $\chi(X)$, $\chi(Y)$ are the Euler characteristics of X and Y respectively and i_*, j_* are the maps on the Whitehead groups induced by

$$\begin{aligned} i\colon X &\longrightarrow X \times Y, \\ x &\mapsto (x, y_0) \end{aligned}$$

$$\begin{aligned} y\colon Y &\longrightarrow X \times Y, \\ y &\mapsto (x_0, y) \end{aligned}$$

for some fixed base points $x_0 \in X$, $y_0 \in Y$.

(5) f *homeomorphism* $\Rightarrow \tau(f) = 0$.

Sketch of proof. (1) After lifting the cellular pushouts to the universal covering, we obtain the following diagram of chain complexes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_*(\widetilde{X}_0) & \longrightarrow & C_*(\widetilde{X}_1) \oplus C_*(\widetilde{X}_2) & \longrightarrow & C_*(\widetilde{X}) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle C_*(\widetilde{f}_0)} & & \big\downarrow{\scriptstyle (C_*(\widetilde{f}_1), C_*(\widetilde{f}_2))} & & \big\downarrow{\scriptstyle C_*(\widetilde{f})} & & \\ 0 & \longrightarrow & C_*(\widetilde{Y}_0) & \longrightarrow & C_*(\widetilde{Y}_1) \oplus C_*(\widetilde{Y}_2) & \longrightarrow & C_*(\widetilde{Y}) & \longrightarrow & 0 \end{array}$$

One can easily check (by constructing a diagram similar to the one above and by applying Lemma 1.11) that $\tau((C_*(\widetilde{f}_1), C_*(\widetilde{f}_2))) = \tau(C_*(\widetilde{f}_1)) + \tau(C_*(\widetilde{f}_2))$. We want to apply Lemma 1.11 (1) to the diagram above, but we need the chain complexes and the chain maps to be defined over the same ring ($\mathbb{Z}\pi$, with $\pi := \pi_1(Y)$). We use, then, the following fact:

Let $f\colon S \to R$ be a ring homomorphism; if (C, ∂) is a finite, based, contractible S-chain complex, then $(C \otimes_S R, \partial \otimes id_R)$ is a finite, based, contractible R-chain complex such that

$$\tau(C \otimes_S R) = f_* \tau(C) \qquad (1)$$

where $f_* : \widetilde{K}_1(S) \to \widetilde{K}_1(R)$ is the map induced by f.

Applying $- \otimes \mathbb{Z}\pi$ to the diagram above, we have:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_*(\widetilde{X}_0) \otimes_{\mathbb{Z}\pi_1(Y_0)} \mathbb{Z}\pi & \longrightarrow & (C_*(\widetilde{X}_1) \otimes_{\mathbb{Z}\pi_1(Y_1)} \mathbb{Z}\pi) \oplus (C_*(\widetilde{X}_2) \otimes_{\mathbb{Z}\pi_1(Y_2)} \mathbb{Z}\pi) & \longrightarrow & C_*(\widetilde{X}) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle C_*(\widetilde{f}_0) \otimes \mathrm{id}_{\mathbb{Z}\pi}} & & \big\downarrow{\scriptstyle (C_*(\widetilde{f}_1) \otimes \mathrm{id}_{\mathbb{Z}\pi}, C_*(\widetilde{f}_2) \otimes \mathrm{id}_{\mathbb{Z}\pi})} & & \big\downarrow{\scriptstyle C_*(\widetilde{f})} & & \\ 0 & \longrightarrow & C_*(\widetilde{Y}_0) \otimes_{\mathbb{Z}\pi_1(Y_0)} \mathbb{Z}\pi & \longrightarrow & (C_*(\widetilde{Y}_1) \otimes_{\mathbb{Z}\pi_1(Y_1)} \mathbb{Z}\pi) \oplus (C_*(\widetilde{Y}_2) \otimes_{\mathbb{Z}\pi_1(Y_2)} \mathbb{Z}\pi) & \longrightarrow & C_*(\widetilde{Y}) & \longrightarrow & 0 \end{array}$$

One can show that this satisfies the conditions of Lemma 1.11 (1). Then:

$$\tau(C_*(\widetilde{f})) = \tau(C_*(\widetilde{f}_1) \otimes \mathrm{id}_{\mathbb{Z}\pi}) + \tau(C_*(\widetilde{f}_2) \otimes \mathrm{id}_{\mathbb{Z}\pi}) - \tau(C_*(\widetilde{f}_0) \otimes \mathrm{id}_{\mathbb{Z}\pi})$$

and by applying 1, the statement follows.

(2) it follows from Lemma 1.11 (2).

(3) it follows from Lemma 1.11 (3) and from an argument similar to the one used to prove part (1).

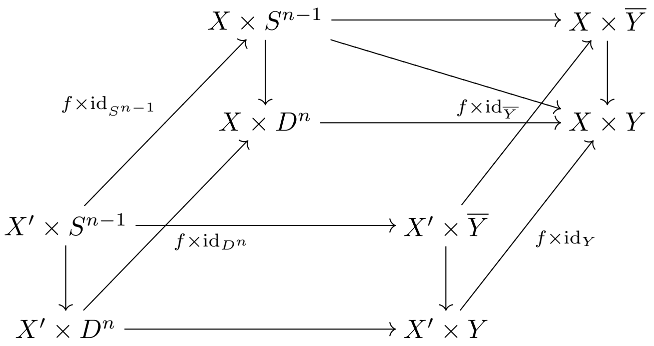
(4) we can write $f \times g = (f \times \mathrm{id}_Y) \circ (\mathrm{id}_X \times g)$ and, by part (3), we have that $\tau(f \times g) = (f \times \mathrm{id}_Y)_* \tau(\mathrm{id}_X \times g) + \tau(f \times \mathrm{id}_Y)$. So it suffices to prove the statement for $f \times \mathrm{id}_Y$ (i.e. for $g = \mathrm{id}_Y$). Notice that, since $\tau(\mathrm{id}) = 0$, we have to show that:

$$\tau(f \times \mathrm{id}_Y) = \chi(Y) i_* \tau(f)$$

We proceed by induction over the cells of Y.

Suppose $Y = \{*\}$, then $\chi(Y) = 1$ and $f \simeq f \times \mathrm{id}_Y$. So $\tau(f \times \mathrm{id}_Y) = i_*\tau(f) = \tau(f)$.

Suppose the statement is true for some $\overline{Y}$, we want to prove it for Y, obtained from $\overline{Y}$ by attaching a one-cell. We have the following pushouts diagrams:



Now one can apply part (1) of the Theorem and conclude after proving the statement for $f \times \mathrm{id}_{S^{n-1}}$ and for $f \times \mathrm{id}_{D^n}$ (Exercise), and observing that $\chi(\bar{Y}) + \chi(D^n) - \chi(S^{n-1}) = \chi(\overline{Y}) + 1 = \chi(Y)$.

(5) this is a deep result due to Chapman.

□

Definition 1.15. Let (W, M_0, d_0, M_1, d_1) be an h-cobordism, define

$$\tau(W, M_0) := (i_0 \circ f_0)_*^{-1}\tau(i_0 \circ f_0) \in \mathrm{Wh}(\pi_1 M_0).$$

Lemma 1.16.

(1) *Let (W, M_0, d_0, M_1, d_1), and $(W', M_0', d_0', M_1', d_1')$ be h-cobordisms and $g\colon M_1 \xrightarrow{\simeq} M_0'$ a diffeomorphism. We have*

$$\tau(W \cup_g W', M_0) = \tau(W, M_0) + \left((i_0 f_0)^{-1}\right)_* (i_1 f_1)_* g_*^{-1}\tau(W', M_0'),$$

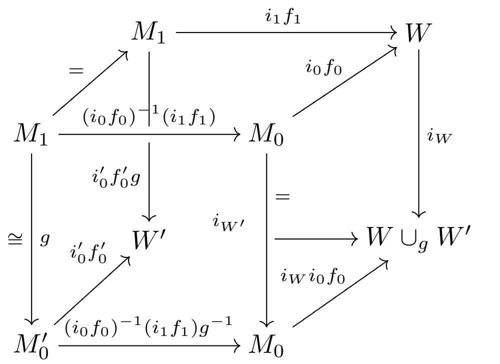
(2) *Let $*$ be the involution induced on the Whitehead group by the canonical involution*

$$\begin{gathered} *\colon \mathbb{Z}\pi \longrightarrow \mathbb{Z}\pi, \\ \sum_{\gamma\in\pi} \lambda_\gamma \gamma \mapsto \sum_{\gamma\in\pi} w_1(\gamma)\lambda_\gamma\gamma^{-1}. \end{gathered}$$

Then

$$*\left(\tau(W, M_0)\right) = (-1)^{\dim M_0}(i_1 f_1)_*^{-1}(i_0 f_0)_*\tau(W, M_1).$$

Proof of (1). The following diagram commutes up to homotopy (front and back pushouts)



Hence applying Theorem 1.11 finishes the proof after a short calculation.

□

Proof of s-cobordism theorem.

Assume W is in normal form, so for some $q \in \mathbb{N}$ the handlebody chain complex $C_*(\widetilde{W}, \widetilde{\partial_0 W}) := H_*(\widetilde{W_*}, \widetilde{W_{*-1}})$ introduced in Chapter 1.2 reduces to

$$0 \longrightarrow C_q^{\text{hb}}(\widetilde{W}, \widetilde{\partial_0 W}) \xrightarrow{\sigma} C_{q-1}^{\text{hb}}(\widetilde{W}, \widetilde{\partial_0 W}) \longrightarrow 0.$$

As in the last talk, let A be the matrix corresponding to the isomorphism σ w.r.t. the chosen bases. By Lemma 1.27, we will only have to show, that $[A]$ and $\tau(W, \partial_0 M)$ correspond nicely. By definition, we have

$$[A] = \tau(C_*^{\text{hb}}(\widetilde{W}, \widetilde{\partial_0 W})).$$

In Chapter 1.2 of his script, Lueck inductively constructs for a given handlebody decomposition of W a CW-Complex X relative $\partial_0 W$ and a homotopy equivalence $f\colon (W, \partial_0 W) \longrightarrow (X, \partial_0 W)$ such that :

- $(X, \partial_0 W)$ has exactly one p cell for each p-handle of $(W, \partial_0 W)$.
- $C_*(f)\colon C_*^{\text{hb}}(\widetilde{W}, \widetilde{\partial_0 W}) \to C_*^{\text{cell}}(\widetilde{X}, \widetilde{\partial_0 W})$ is a based chain complex isomorphism.
- $f_*\colon C_*^{\text{cell}}(\widetilde{W}, \widetilde{\partial_0 W}) \to C_*^{\text{cell}}(\widetilde{X}, \widetilde{\partial_0 W})$ is a simple (i.e $\tau(f) = 0$) chain homotopy equivalence. To see this, you have to inspect the inductive construction of f via pushout maps and apply the calculation of Whitehead torsion for cellular pushout diagrams.

By the second remark, we have $[A] = \tau(\widetilde{X}, \widetilde{\partial_0 W})$, by using the third remark and the corresponding results for $[A]$ we can thus proof part (i) and (ii) of the s-cobordism theorem.

Part (iii) of the s-cobordism theorem follows from part (i) and (ii) and our previous lemma. Assume that (W, M_0, M_1) and (W', M_0, M'_1) are h-cobordism over M_0 and $\tau(W, M_0) = \tau(W', M_0)$. By part (ii) of the s-cobordism theorem, we can find an h-cobordism (W'', M_1, M''_2) over M_1 such that $\tau(W'', M_1) = -(i_1 f_1)_*^{-1}(i_0 f_0)_* \tau(W, M_0)$.

If we glue W and W'' along M_1, by the previous lemma we get $\tau(W \cup_{M_1} W'', M_0) = 0$. By the first part of the s-cobordism theorem we therefore get a diffeomorphism $M_0 \times I \to (W \cup_{M_1} W'', M_0)$. By restricting this map to $M_0 \times \{1\}$ we receive a diffeomorphism $g\colon M_0 \to M''_2$ and thus can glue W' along M_0 to $W \cup_{M_1} W''$. The same arguments applied to $W' \cup_g W''$ leads to a diffeomorphism $W' \cup_g W'' \cong M_1 \times I$. In total we thus get a diffeomorphism $(W, M_0) \cong (W', M_0)$ (compare figure 1).

□

Figure 1. Proof of part (iii) of the s-cobordism theorem