1. Manifold Atlas : Regensburg Surgery Blockseminar 2012
The s-Cobordism Theorem I (Farid Madani and Mihaela Pilca)

Definition 1.1.
- An *n-dimensional cobordism* (W, M_0, f_0, M_1, f_1) is a 5-tuple where W is a compact n-dimensional manifold with boundary $\partial W = \partial_0 W \amalg \partial_! W$ and $f_0 : M_0 \xrightarrow{\simeq} \partial_0 W$, $f_1 : M_1 \xrightarrow{\simeq} \partial_1 W$ diffeomorphisms preserving the orientation.
- Two cobordisms $(W^i, M_0, f_0^i, M_1^i, f_1^i), i = 0, 1$, are *diffeomorphic rel. to* M_0 if there exists $F : W^0 \to W^1$ such that $F \circ f_0^0 = f_0^1$.
- (W, M_0, f_0, M_1, f_1) is an *h-cobordism* iff the inclusions $\partial_i W \hookrightarrow W$ are homotopy equivalences for $i = 0, 1$.
- (W, M_0, f_0, M_1, f_1) is *trivial* if it is diffeomorphic to the trivial h-cobordism
$$(M_0 \times [0, 1], M_0 \times \{0\}, M_0 \times \{1\}).$$

Theorem 1.2 (s-cobordism Theorem)**.** *Let M_0 be a closed, connected, oriented manifold of dim $n \geq 5$ with $\pi_1(M_0) = \pi$. The followings hold:*

(1) *if (W, M_0, f_0, M_1, f_1) is an h-cobordim over M_0, then W is trivial over M_0 iff its Whitehead torsion $\tau(W, M_0) \in \mathrm{Wh}(\pi)$ vanishes.*
(2) *For any $x \in \mathrm{Wh}(\pi)$ there is an h-cobordism (W, M_0, f_0, M_1, f_1) over M_0 with $\tau(W, M_0) = x$.*
(3) *The map $\{(W, M_0, f_0, M_1, f_1)$h-cobordism$\} \to \tau(W, M_0)$ yields a bijection*
$$\{\textit{diffeo classes rel. to } M_0 \textit{ of h-cobordism over } M_0\} \to \mathrm{Wh}(\pi).$$

Applications.

h-cobordism Theorem. Every h-cobordism over a simply connected manifold M_0, dim $M_0 \geq 5$ is trivial.

Poincaré conjecture. If M is a closed simply connected manifold, $\dim(M) \geq 5$ and M has the homology of the n-sphere, then M is homeomorphic to S^n.

Proof. (sketch) For dim $M \geq 6$, consider $W := M - (D_0^n \amalg D_1^n)$ (M with 2 embedded disjoint disks removed). Prove that W is an h-cobordism and apply the h-cobordism theorem. ☐

Handlebody decomposition.

Definition 1.3. An *n-dimensional q-handle* is $D^q \times D^{n-q}$. Its *core* is $D^q \times \{0\}$, its *cocore* $\{0\} \times D^{n-q}$, its *transverse sphere* is $\{0\} \times S^{n-q-1}$.

Let $(M, \partial M)$ be an n-dimensional manifold with boundary and let $\varphi^q : S^{q-1} \times D^{n-q} \to \partial M$ be an embedding. Define:
$$M + (\varphi^q) := M \cup_{\varphi^q} D^q \times D^{n-q}.$$
The boundary of the new manifold is then given by
$$\partial(M + (\varphi^q)) = (\partial M - \mathrm{int}(\varphi^q(S^{q-1} \times D^{n-q}))) \cup_{\varphi^q|_{S^{q-1}\times S^{n-q-1}}} D^q \times S^{n-q-1}.$$
From now on $(W, \partial_0 W, \partial_1 W)$ denotes an n-dimensional cobordism with $\partial W = \partial_0 W \amalg \partial_1 W$.

Handlebody Decomposition Lemma.
$$W \simeq \partial_0 W \times [0, 1] + \sum_{i=1}^{p_0} (\varphi_i^0) + \cdots + \sum_{i=1}^{p_n} (\varphi_i^n).$$

Lemma 1.4 (Cancellation Lemma)**.** *Let* $\varphi^q : S^{q-1} \times D^{n-q} \to \partial_1 W$ *be an embedding and* $\psi^{q+1} : S^q \times D^{n-1-q} \to \partial_1(W + (\varphi^q))$ *be another embedding. Suppose that* $\psi^{q+1}(S^q \times \{0\})$ *is transversal to the transverse sphere of the handle* (φ^q) *and meets it in exactly one point. Then there exists a diffeomorphism rel.* $\partial_0 W$ *from* W *to* $W + (\varphi^q) + (\psi^{q+1})$.

Notation.

$$
\begin{aligned}
W_q &= \partial_0 W \times [0,1] + \sum_{i=1}^{p_0} (\varphi_i^0) + \cdots + \sum_{i=1}^{p_q} (\varphi_i^q), \\
W_{-1} &= \partial_0 W \times [0,1], W_n = W, \\
\partial_1 W_q &= \partial W_q - \partial_0 W \times \{0\}, \\
\partial_1^0 W_q &= \partial_1 W_q - \coprod_{i=1}^{p_{q+1}} \varphi^{q+1}(S^q \times \mathrm{int}(D^{n-1-q})) \subset \partial_1 W_{q+1}.
\end{aligned}
$$

Working Definition. $\psi : S^q \to \partial_1 W_q$ is called (q, i_0)-*transversal embedding* if $\psi(S^q)$ meets the transversal sphere of $(\varphi_{i_0}^q)$ transversally at exactly one point and is disjoint from the transversal spheres of all the other handles (φ_i^q), for $i \neq i_0$.

Lemma 1.5 (Elimination Lemma)**.** *Fix* q *with* $1 \leq q \leq n-3$. *Suppose that* $p_j = 0$ *for* $j < q$. *Fix* $1 \leq i_0 \leq p_q$. *Suppose that there exists an embedding* $\psi^{q+1} : S^q \times D^{n-q-1} \to \partial_1^0 W_q$ *satisfying:*

1. $\psi^{q+1}|_{S^q \times \{0\}}$ *is isotopic in* $\partial_1 W_q$ *to a* (q, i_0)*-transversal embedding* $\psi_1^{q+1} : S^q \times \{0\} \to \partial_1^0 W_q$.
2. $\psi^{q+1}|_{S^q \times \{0\}}$ *is isotopic in* $\partial_1 W_{q+1}$ *to a trivial embedding* $\psi_2^{q+1} : S^q \times \{0\} \to \partial_1 W_{q+1}$ *(i.e. there exists an embedding* $\Psi_1 : D^{n-1} \to \partial_1 W_{q+1}$ *such that* $\psi_1^{q+1} = \Psi_1 \circ \Psi_2|_{S^q \times \{0\}}$, *where* $\Psi_2 : S^q \times D^{n-q-1} \to D^{n-1}$ *is a fixed standard embedding).*

Then W *is diffeomorphic rel.* $\partial_0 W$ *to*

$$
\partial_0 W \times [0,1] + \sum_{\substack{i=0 \\ i \neq i_0}}^{p_q} (\varphi_i^q) + \sum_{i=0}^{p_{q+1}} (\bar{\varphi}_i^{q+1}) + (\psi^{q+2}) + \sum_{i=0}^{p_{q+2}} (\bar{\varphi}_i^{q+2}) \cdots + \sum_{i=0}^{p_n} (\bar{\varphi}_i^n).
$$

Let $p : \widetilde{W} \to W$ be the universal covering of W. Then $p : \widetilde{W}_q := p^{-1}(W_q) \to W_q$ is the universal covering of W_q for $q \geq 2$. We define the chain complex

$$
C_*(\widetilde{W}, \partial_0 \widetilde{W}) : C_q(\widetilde{W}, \partial_0 \widetilde{W}) := H_q(\widetilde{W}_q, \widetilde{W}_{q-1}), \text{with } d_q = i_q \circ \partial_q, \text{where:}
$$

$$
H_q(\widetilde{W}_q, \widetilde{W}_{q-1}) \xrightarrow{\partial_q} H_q(\widetilde{W}_{q-1}) \xrightarrow{i_q} H_{q-1}(\widetilde{W}_{q-1}, \widetilde{W}_{q-2}).
$$

The characteristic map

$$
(\Phi_i^q, \varphi_i^q) : (D^q \times D^{n-q}, S^{q-1} \times D^{n-q}) \to (W_q, W_{q-1})
$$

induces the following class

$$
[(\varphi_i^q)] := H_q(\tilde{\Phi}_i^q, \tilde{\varphi}_i^q)[D^q] \in H_q(\widetilde{W}_q, \widetilde{W}_{q-1}), q \geq 2.
$$

It turns out that $\{[(\varphi_i^q)] \,|\, 1 \leq i \leq p_q\}$ is a $\mathbb{Z}\pi$-basis of $C_q(\widetilde{W}, \partial_0 \widetilde{W})$. Fix $z \in \partial_0 W, \tilde{z} \in \partial_0 \widetilde{W}$ such that $p(\tilde{z}) = z$. If W has no handles of index 0 and 1, then

$\pi_q(\widetilde{W}_q, \widetilde{W}_{q-1}) \xrightarrow{\partial_q} \pi_{q-1}(\widetilde{W}_{q-1}) \xrightarrow{\pi_{q-1}(i_q)} \pi_{q-1}(\widetilde{W}_{q-1}, \widetilde{W}_{q-2})$, $\pi_q(W_q, W_{q-1}) = 0$ for $q = 0, 1$. Using Hurewicz isomorphism, we get $\pi_q(W_q, W_{q-1}) \simeq H_q(\widetilde{W}_q, \widetilde{W}_{q-1})$. Seen through this isomorphism, the class $[(\varphi_i^q)] \in \pi_q(W_q, W_{q-1})$ is represented by $(\Phi_i^q, \varphi_i^q) : (D^q \times \{0\}, S^{q-1} \times \{0\}) \to (W_q, W_{q-1})$.

From now on $(W, \partial_0 W, \partial_1 W)$ is an h-cobordism.

Lemma 1.6 (Homology Lemma)**.** *Suppose* $n \geq 6$. *Fix* $2 \leq q \leq n-3$ *and* $i_0 \leq p_q$. *Let* $f : S^q \to \partial_1 W_q$ *be an embedding,* $\widetilde{f} : S^q \to \widetilde{W}_q$ *a lift of* f *to* $\widetilde{W}_q$ *and* $[\widetilde{f}]$ *the image of the class represented by* $\widetilde{f}$ *in* $H_q(\widetilde{W}_q, \widetilde{W}_{q-1})$ *obtained by* $\pi_q(\widetilde{W}) \to \pi_q(\widetilde{W}_q, \widetilde{W}_{q-1}) \xrightarrow{\simeq} H_q(\widetilde{W}_q, \widetilde{W}_{q-1})$. *The following statements are equivalent:*

(1) f *is isotopic to a* (q, i_0)*-transversal embedding* $g : S^q \to \partial_1 W_q$.

(2) *There exists* $\gamma \in \pi$ *such that* $[\widetilde{f}] = \pm\gamma[(\varphi^q_{i_0})]$.

Lemma 1.7 (Modification Lemma)**.** *Let* $f : S^q \to \partial_1^0 W_q$ *be an embedding and let* $x_j \in \mathbb{Z}\pi$ *for* $j = 1, \ldots, p_{q+1}$. *Then there exists an embedding* $g \colon S^q \to \partial_1^0 W_q$ *such that:*

(1) f *and* g *are isotopic in* $\partial_1 W_{q+1}$,

(2) *for a given lift* $\widetilde{f} : S^q \to \widetilde{W}_q$ *of* f *there exists a lift* $\tilde{g} : S^q \to \widetilde{W}_q$ *of* g *such that:*

$$[\tilde{g}] = [\widetilde{f}] + \sum_{j=1}^{p_{q+1}} x_j d_{q+1}[(\varphi_j^{q+1})] \quad \textit{in } \pi_q(\widetilde{W}_q, \widetilde{W}_{q-1}) \simeq H_q(\widetilde{W}_q, \widetilde{W}_{q-1}).$$

Proof. W.l.o.g. it is sufficient to prove that $[\tilde{g}] = [\widetilde{f}] + \varepsilon\gamma d_{q+1}[(\varphi_j^{q+1})]$, for $\varepsilon = \pm 1$ and $\gamma \in \pi$. Set $g = f \#_w t_j$ (see figure) and let $\tilde{g}$ be the unique lift of g such that $\widetilde{f} = \tilde{g}$ for all points where $f = g$. Therefore $[\tilde{g}] = [\widetilde{f}] + \gamma[\tilde{t_j}]$, with $[\tilde{t}_j] = [t_j] = [\varphi_j^{q+1}|_{S^q \times \{0\}} \colon S^q \to W_q] = d_{q+1}[(\varphi_j^{q+1})]$. □