Lecture 2 - Normal maps and surgery below the middle dimension

Spiros Adams-Florou

Westfälische Wilhelms-Universität Münster

26th August, 2013

The surgery Programme

Notation

Manifolds are understood to be smooth compact and closed unless stated otherwise. M^m denotes M is m-dimensional.

Remark

Unknown in general if $\pi_1(M) \cong \pi_1(M') \Rightarrow$ *unknown in general if* $M \cong M'$.

The surgery programme attempts to answer

(A): *Given* $f : M^m \xrightarrow{\simeq} (M')^m$, *is* $f \sim$*diffeomorphism?*
(B): *Given* X *with* m*-dimensional Poincaré duality, is* X *homotopy equivalent to an* m*-dimensional manifold?*

(A) is relative (B).

Definition

An m-**dimensional geometric Poincaré complex** (m-gPc) is a finite CW-complex X with an orientation character $\omega \in H^1(X;\mathbb{Z}_2)$ and ω-twisted fundamental class $[X] \in H_m(X;\mathbb{Z}^\omega)$ s.t.

$$[X] \cap - : H^*(\widetilde{X}) \to H_{m-*}(\widetilde{X})$$

are $\mathbb{Z}[\pi_1(X)]$-module isomorphisms.

Definition

Let X be an m-gPc.

(i) A **manifold structure** (M, f) on X is an m-dimensional manifold M together with a homotopy equivalence $f : M \to X$.

(ii) The **manifold structure set** $\mathcal{S}(X)$ of X is the set of equivalence classes of manifold structures (M, f) where $(M, f) \sim (M', f')$ if there exists a bordism $(F, f, f') : (W, M, M') \to X \times (I, \{0\}, \{1\})$ with F a homotopy equivalence so that (W, M, M') is an h-cobordism.

Thus we are asking

Let X be an m-gPc,
(A'): Given (M, f), $(M', f') \in \mathcal{S}(X)$, does $(M, f) \sim (M', f')$?
(B'): Is $\mathcal{S}(X)$ non-empty?

Definition

(i) Let X be a connected m-gPc with a k-dimensional vector bundle $\xi : E \to X$. A **normal k-map** $(M, i, f, \overline{f})$ consists of a closed m-dimensional manifold M together with an embedding $i : M \to \mathbb{R}^{m+k}$ and a bundle map $\overline{f} : \nu_M = \nu(i) \to \xi$ that is a fibrewise isomorphism covering a map $f : M \to X$, I.e. $\nu_M \cong f^*(\xi)$. Shorten this to $(\overline{f}, f) : M^m \to X$.

(ii) We call a normal k-map $(\overline{f}, f)$ a **degree one normal k-map** if f has degree one, i.e. $f_*[M] = [X] \in H_m(X)$.

(iii) A **normal bordism** is a normal k-map from a cobordism:

$$((\overline{F}, F), (\overline{f}, f), (\overline{f}', f')) : (W, M, M') \to X \times (I, \{0\}, \{1\}).$$

Normal bundles vs Tangent bundles

<u>Normal data</u>

$$\begin{array}{ccc} \nu_M & \xrightarrow{\bar{f}} & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

$$\nu_M \cong f^*\xi.$$

<u>Tangential data</u>

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}}^a & \xrightarrow{\bar{f}'} & \xi^{-1} \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

$$TM \oplus \underline{\mathbb{R}}^a \cong f^*\xi^{-1}.$$

Remark

For X an m-gPc, $\mathcal{S}(X)$ non-empty only if there exists a degree one normal map (d1nm) with target X.

Proof.

Given $f : M \xrightarrow{\sim} X$, choose a homotopy inverse f^{-1} and $h : \mathrm{id}_M \simeq f^{-1} \circ f$. Lift $h : M \times I \to M$ to a bundle map $\overline{h} : \nu_{M\times I} = \nu_M \times I \to \nu_M$ s.t. $\overline{h}|_{\nu_M \times \{0\}} = \mathrm{id}_{\nu_M}$. $\overline{h}|_{\nu_M \times \{1\}} : \nu_M \to \nu_M$ covers $f^{-1} \circ f : M \to M$ and so induces a bundle map $\overline{f} : \nu_M \to (f^{-1})^* \nu_M$ covering $f : M \to X$. Clearly $f^*(f^{-1})^* \nu_M \cong \nu_M$. □

Remark

We need normal maps!

The surgery method

The surgery method proceeds via a two stage obstruction theory:

- ($B1$): Does an m-gPC X admit a degree one normal map $(\bar{f}, f) : M \to X$?
- ($B2$): If so, is there a d1nm that is bordant to a homotopy equivalence $(\bar{f}', f') : M' \xrightarrow{\sim} X$?

Or for the relative problem

- ($A1$): Are homotopy equivalent m-dimensional manifolds M, M' normally cobordant?
- ($A2$): If so, are they h-cobordant?

Remark

There is a version for non-simply connected spaces taking Whitehead torsion into account - more on this later.

First obstruction

We concentrate on question $(B2)$ but briefly answer $(B1)$.

Recall

Let $O(k)$ be the orthogonal group.

$$O := \lim_{k \to \infty} O(k).$$

BO is the classifying space for vector bundles where

$$BO := \lim_{k \to \infty} BO(k)$$

for $BO(k) = G_k(\mathbb{R}^\infty)$ the Grassmannian of k-planes in $\mathbb{R}^\infty$.

Definition

- Let $G(k)$ denote the monoid of self-homotopy equivalences of S^{k-1}.
$$G := \lim_{k\to\infty} G(k).$$
BG classifies spherical fibrations.
- There are maps
$$J_k : O(k) \to G(k)$$
$$BJ_k : BO(k) \to BG(k)$$
so taking colimits we get maps $J : O \to G$ and $BJ : BO \to BG$.
- Let G/O denote the mapping fibre of $BJ : BO \to BG$ so that there is a fibration (up to homotopy)
$$G/O \to BO \to BG$$
and hence a long exact sequence in homotopy groups.

- All manifolds M^m have a (stable) normal vector bundle $\nu_M : M \to BO$,
- All m-gPc's X have a **Spivak normal spherical fibration** (SNF) $\nu_X : X \to BG$,
- An m-gPc admits a degree one normal map if and only if its SNF has a vector bundle reduction, i.e. $\nu_X : X \to BG$ lifts to a map to BO, if and only if $t(\nu_X) \simeq *$ for

$$\begin{array}{ccc} & & BO \\ & \nearrow & \downarrow{\scriptstyle BJ} \\ X & \xrightarrow{\nu_X} & BG \\ & {\scriptstyle t(\nu_X)} \searrow & \downarrow \\ & & B(G/O). \end{array}$$

Strategy

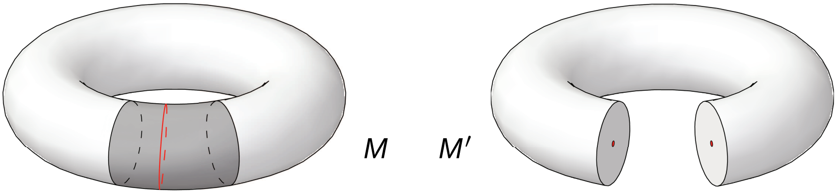
- Start with a degree one normal map $(\bar{f}, f) : M \to X$,
- By Whitehead's theorem, f is a homotopy equivalence if and only if $\pi_*(f) = 0$,
- We show that $H_{i+1}(\widetilde{f})$ have Poincaré duality,
- Thus if $\pi_{i+1}(f)$ vanish up to and including the middle dimension then f is a homotopy equivalence,
- Inductively seek to kill $\pi_{i+1}(f)$ by surgery,
- Surgery gives a normal bordism to a new normal map $(\bar{f}', f')$ with (hopefully) that homotopy group killed,
- Thus if we can do surgery to kill $\pi_{i+1}(f)$ up to and including the middle dimension we get a normal bordism to a homotopy equivalence.

Definition

An n**-surgery** ($n \geq -1$) on an m-dimensional manifold M is the procedure of constructing a new manifold

$$M' := \overline{M \backslash S^n \times D^{m-n}} \cup_{S^n \times S^{m-n-1}} D^{n+1} \times S^{m-n-1}$$

by cutting out an embedded $S^n \times D^{m-n} \subset M$ and replacing it by $D^{n+1} \times S^{m-n-1}$. (N.b. $S^{-1} := \emptyset$.)



Remark

- *Poincaré duality is preserved: the **effect of surgery**, M', is also a (smooth) manifold.*
- *Homotopically an n-surgery kills the class of $S^n \times D^{m-n} \in \pi_n(M)$ but also creates the (dual) class of $D^{n+1} \times S^{m-n-1} \in \pi_{m-n-1}(M')$.*
- *An n-surgery gives a (smooth) cobordism (W, M, M') called the **trace** of the surgery. W is obtained by attaching an $(n+1)$-**handle**, $D^{n+1} \times D^{m-n}$, to $M \times I$ at $S^n \times D^{m-n} \times \{1\} \subset M \times \{1\}$.*

Proposition

Two smooth m-dimensional manifolds M, M' are cobordant if and only if M' can be obtained from M by a finite sequence of surgeries.

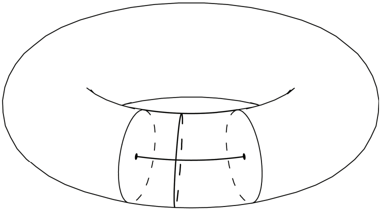


Figure: Our example

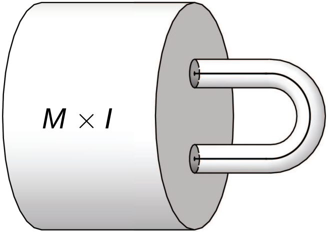


Figure: General picture

Remark

*For an element $x \in \pi_n(M)$, we can do surgery to kill x if and only if x may be represented by a **framed** embedding $\overline{g} : S^n \times D^{m-n} \hookrightarrow M$ if and only if we can represent x by an embedding $g : S^n \hookrightarrow M$ with trivial normal bundle.*

Definition

Recall, the **relative homotopy groups** $\pi_{n+1}(f)$ $(n \geqslant 0)$ of a pointed map $f : M \to X$ are designed to fit into a long exact sequence

$$\cdots \longrightarrow \pi_{n+1}(M) \xrightarrow{f_*} \pi_{n+1}(X) \longrightarrow \pi_{n+1}(f) \longrightarrow \pi_n(M) \longrightarrow \cdots.$$

$x \in \pi_{n+1}(f)$ represented by homotopy classes of commuting diagrams

$$\begin{array}{ccc} S^n & \xrightarrow{g} & M \\ \downarrow & & \downarrow f \\ D^{n+1} & \xrightarrow{h} & X \end{array}$$

with $g : S^n \to M$, $h : D^{n+1} \to X$ pointed maps.

Theorem (Whitney immersion theorem)

For $2n \leqslant m$ every map $f : N^n \to M^m$ is homotopic to an immersion $N \looparrowright M$, and for $2n + 1 \leqslant m$ any two homotopic immersions are regular homotopic (homotopic through immersions).

Theorem (Whitney embedding theorem)

(i) *For $2n + 1 \leqslant m$ every map $N^n \to M^m$ is homotopic to an embedding $N \hookrightarrow M$, and for $2n + 2 \leqslant m$ any two homotopic embeddings are isotopic (homotopic through embeddings).*

(ii) *For $n \geqslant 3$ and $\pi_1(M) = \{1\}$ every map $f : N^n \to M^{2n}$ is homotopic to an embedding $N \hookrightarrow M$.*

Corollary

Below the middle dimension, $2n+1 \leqslant m$, all $x \in \pi_{n+1}(f : M^m \to X)$ can be represented by embeddings. Can we frame?

Proposition

Let $2n \leqslant m$ and let $x \in \pi_{n+1}(f : M^m \to X)$ be represented

$$\begin{array}{ccc} S^n & \xrightarrow{g} & M \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\ D^{n+1} & \xrightarrow{h} & X \end{array}$$

with g an immersion. We may extend this to a framed immersion.

for $2n+1 \leq m$.

- The normal data contained in a normal map $(\bar{f}, f) : M^n \to X$ forces the pullback $g^*\nu_M$ to be trivial:

$$g^*\nu_M \cong g^*f^*\xi \cong j^*\underline{\mathbb{R}}^a = \underline{\mathbb{R}}^a.$$

- $\nu(g) \oplus TS^n \cong g^*TM$ so

$$\nu(g) \oplus TS^n \oplus \underline{\mathbb{R}}^a \cong g^*(TM \oplus \nu_M) \cong \underline{\mathbb{R}}^b.$$

- Since $TS^n \oplus \underline{\mathbb{R}} \cong \underline{\mathbb{R}}^{n+1}$ we get that $\nu(g)$ is stably trivial.
- $\nu(g)$ correponds to a map $S^n \to BO(m-n)$, i.e. an element of $\pi_n(BO(m-n))$. This element is sent to zero under the inclusion $BO(m-n) \to BO(m-n+(n+a))$.
- **Fact:** The pair $(BO(n+2), BO(n+1))$ is $(n+1)$-connected so $\pi_{n+1}(BO(n+1)) \cong \pi_{n+1}(BO(n+1+a))$ and hence $\nu(g)$ is trivial meaning we can frame the immersion.

□

Surgery on a normal map

Let $(\bar{f}, f) : M^m \to X$ be a normal map and suppose $x \in \pi_{n+1}(f)$ is represented by a framed embedding

$$\begin{array}{ccc} S^n \times D^{m-n} & \xrightarrow{g} & M \\ \downarrow & & \downarrow f \\ D^{n+1} \times D^{m-n} & \xrightarrow{h} & X \end{array}$$

Define $f' : M' \to X$ by

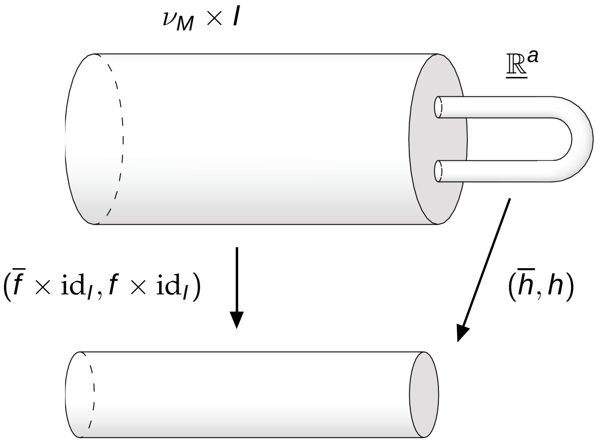
$$f' = f| \cup h| : \overline{M \backslash S^n \times D^{m-n}} \cup D^{n+1} \times D^{m-n} \to X.$$

f and f' are bordant:

$$F = f \times \mathrm{id}_I \cup h : W = M \times I \cup D^{n+1} \times D^{m-n} \to X \times I \cup X \times \{1\}.$$

We can extend the normal data over the bordism:

- Cover $f \times \mathrm{id}_I$ with $\overline{f} \times \mathrm{id}_I$,
- Cover $h : D^{n+1} \times D^{m-n} \to X \times \{1\}$ with $\overline{h} : h^*(\xi) = \underline{\mathbb{R}}^a \to \xi$,
- Compatible since $g^*(\nu_M)$ trivial and $h = f$ on $S^n \times D^{m-n}$.



Surgery below the middle dimension

Corollary

We can embed and frame any homotopy class $x \in \pi_{n+1}(f)$ below the middle dimension for a normal map f, and surgery on the framed embedding gives a normal bordism to a new normal map.

Let $(\bar{f}, f) : M^m \to X$ be a d1nm.

- First kill off $\pi_1(f)$ with 0-surgeries.
- Next kill off $\pi_2(f)$ by construction of Kreck:
  M compact, X finite so both have finitely presented π_1:

$$\pi_1(X) = \langle y_1, \ldots, y_k | r_1, \ldots, r_s \rangle$$
$$\pi_1(M) = \langle a_1, \ldots, a_j | R_1 \ldots, R_r \rangle$$

Perform k 0-surgeries on M resulting in

$$M' = M \sharp \mathop{\sharp}_{i=1}^{k} (S^1 \times D^{m-1})_i.$$

Now

$$\pi_1(M) = \langle a_1, \ldots, a_j, z_1, \ldots, z_k | R_1 \ldots, R_r \rangle$$

and we can choose the surgeries so that $(f')_*(z_i) = y_i$ e.g. by choosing the nullhomotopies $h : D^1 \times D^{m-1}$ in X to map along the loop x_i.
Consider $(f')_*(a_i)$. This is some word $w_i(y_1, \ldots, y_k) \in \pi_1(X)$. Hence

$$a_i^{-1} w_i(z_i, \ldots, z_k) \quad \text{and} \quad r_i(z_i, \ldots, z_k)$$

are both in $\ker(f')_*$.
We can do surgery on these loops resulting in M''. New π_1 is old with these words added to relations. Can check this is now π_1-isomorphism.

Suppose that f is n-connected for $n \geqslant 1$ so that $\pi_1(M) \cong \pi_1(X) =: \pi$.

Definition

Let $\widetilde{X}$ be the universal cover of X and $\widetilde{M} = f^*(\widetilde{X})$ the pullback cover with $\widetilde{f} : \widetilde{M} \to \widetilde{X}$ a π-equivariant lift. We define the **kernel homology and cohomology $\mathbb{Z}[\pi]$-modules of $f : M \to X$** by

$$K_*(M) := H_{*+1}(\widetilde{f}), \quad K^*(M) := H^{*+1}(\widetilde{f}).$$

Since f is n-connected the Hurewicz theorem tells us that

$$K_i(M) = H_{i+1}(\widetilde{f}) \cong \pi_{i+1}(\widetilde{f}) = 0, \quad i < n$$
$$K_n(M) = H_{n+1}(\widetilde{f}) \cong \pi_{n+1}(\widetilde{f}) = \pi_{n+1}(f).$$

Definition

For maps of pairs $(M, \partial M) \to (X, \partial X)$ we have $K_i(M, \partial M)$ and $K^i(M, \partial M)$ defined to fit into corresponding long exact sequences of $\mathbb{Z}[\pi]$-modules.

Proposition

The homology and cohomology kernel modules are related by Poincaré duality isomorphisms:

$$K^*(M) \cong K_{m-*}(M).$$

Proof.

The Poincaré duality isomophism of $\widetilde{M}$ splits

$$H^n(\widetilde{M}) = K^n(M) \oplus H^n(\widetilde{X}) \to H_{m-n}(\widetilde{M}) = K_{m-n}(M) \oplus H_{m-n}(\widetilde{X}).$$

□

Proposition

Suppose $(\bar{f}, f) : M^m \to X$ is an n-connected (degree one) normal map. Then $K_n(M)$ is finitely generated as a $\mathbb{Z}[\pi]$-module (and in the case $m = 2n$ stably f.g. free).

Proposition

Let $(\bar{f}, f) : M^m \to X$ *be an m-dimensional normal k-map and let*

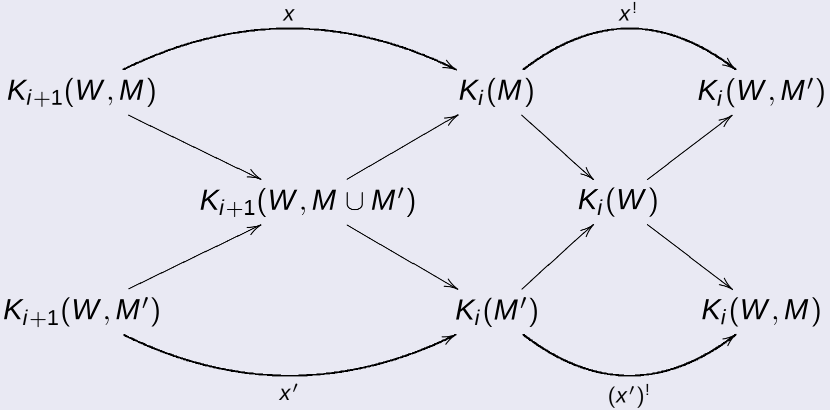
$$(\bar{F}, F) : (W^{m+1}, M, M') \to X \times (I, \{0\}, \{1\})$$

be a normal bordism from the trace of an n-surgery on $(\bar{f}, f)$ *killing an element* $x \in \pi_{n+1}(f)$*, and let* $M_0 = \overline{M \backslash (S^n \times D^{m-n})}$*.*
The kernel $\mathbb{Z}[\pi]$*-modules are such that*

$$K_i(W, M) = \begin{cases} \mathbb{Z}[\pi], & i = n+1 \\ 0, & i \neq n+1, \end{cases}$$

$$K_i(W, M') = \begin{cases} \mathbb{Z}[\pi], & i = m-n \\ 0, & i \neq m-n \end{cases}$$

with a commutative braid of exact sequences of $\mathbb{Z}[\pi]$-modules



s.t.

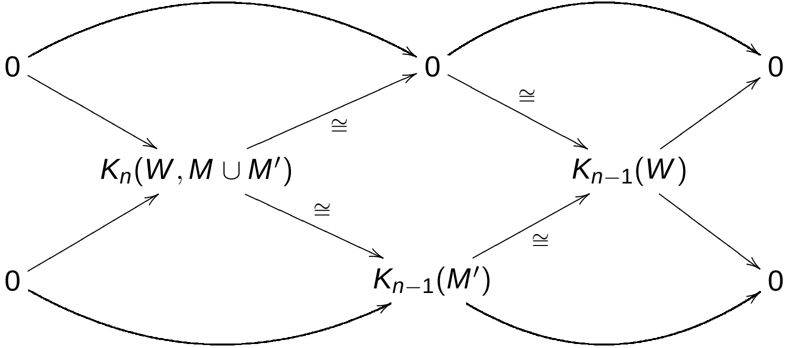
$$x : K_{n+1}(W, M) = \mathbb{Z}[\pi] \to K_n(M); 1 \to x,$$
$$x^! : K_{m-n}(M) \to K_{m-n}(W, M') = \mathbb{Z}[\pi]; y \to \lambda(x, y)$$

where λ denotes the homology intersection form:

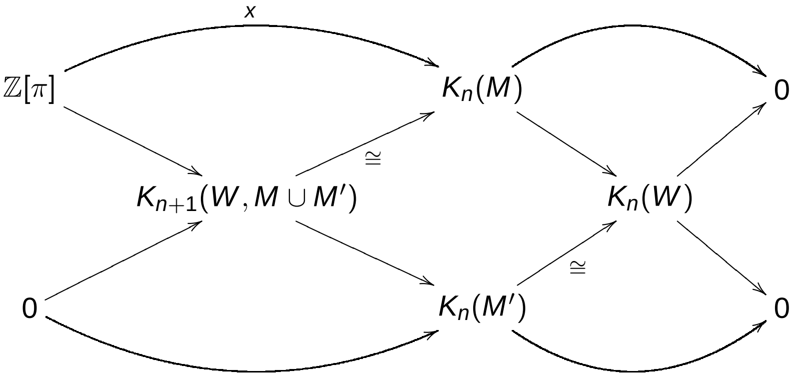
$$\begin{array}{rcl} \lambda^{alg} : K_n(M) \times K_n(M) & \to & \mathbb{Z}[\pi] \\ (x, y) & \mapsto & x^*(y) \end{array}$$

where $x^* \in K^n(M)$ is the Poincaré dual w.r.t. kernel module Poincaré duality.

Let $2n+1 < m$. Suppose $(\bar{f}, f) : M^m \to X$ is an n-connected degree one normal map, and we do surgery on a framed embedding representing $x \in \pi_{n+1}(f)$. Interpreting the braid we see



so $K_{n-1}(M') = 0$.



hence $K_n(M) \cong K_{n+1}(W, M \cup M')$ and

$$\begin{aligned} K_n(M') &\cong K_{n+1}(W, M \cup M') / \mathrm{Im}(\mathbb{Z}[\pi] \to K_n(M)) \\ &\cong K_n(M) / \langle x \rangle . \end{aligned}$$

Thus the generator $x \in K_n(M)$ is killed off in $K_n(M')$.

Corollary

Let $m = 2n$ or $2n + 1$. Then any m-dimensional degree one normal map $(\overline{f}, f) : M \to X$ is normal bordant to an n-connected degree one normal map $(\overline{f}', f') : M' \to X$.

Problems

- *In the middle dimension for $m = 2n$ Whitney's embedding theorem doesn't hold - cannot always represent $x \in \pi_{n+1}(f)$ by embeddings.*
- *In the middle dimension for $m = 2n + 1$ we can still embed and frame, however we shall see that surgery on a framed embedding doesn't necessarily make the surgery kernel $K_n(M)$ any smaller...*