Manifold Atlas : Regensburg Surgery Blockseminar 2012

Fake tori (Anna Abczynski and Philipp Kühl)

1. Motivation and strategy

In this talk we want to give an explicite application of the surgery long exact sequence in the topological category. We want to show

Theorem 1.1. *The topological structure set of the $n-$torus $\mathscr{S}^{TOP}(T^n)$ is trivial, i.e. every space that is homotopy equivalent to a torus is already homeomorphic to a torus.*

Note that this does not hold in the PL-category. There we get so-called fake tori that is spaces that are homotopy equivalent to the torus but not $PL-$isomorphic.

In our situation the surgery exact sequence has the form

$$... \to [\Sigma T^n_+, G/TOP] \xrightarrow{\sigma_1} L_{n+1}(\mathbb{Z}^n) \xrightarrow{\partial} \mathscr{S}^{TOP}(T^n) \xrightarrow{\eta} [T^n, G/TOP] \xrightarrow{\sigma} L_n(\mathbb{Z}^n),$$

where we abbreviate the group ring $\mathbb{Z}[\mathbb{Z}^n]$ by $\mathbb{Z}^n$. Hence to prove the theorem in dimensions ≥ 5 it suffices to show that σ is injective (Lemma 3.1) and σ_1 is onto (Lemma 4.1).

Note that we want to work in the category of topological manifolds, i.p. we a priori do not have concepts such as tangent spaces or transversality. There is, however, a complex theory which introduces substitutes for those. In the end one can work with them in the fashion we are used to. Hence we will use this a a (huge) black box and work as if we had developed this theory.

2. Basics

In order to show the properties of σ and σ_1 we need the Shaneson splitting which we won't state in full generality since we will only need it for a specific case. Hence we allow ourselfs to restrict to groups with trival Whitehead group. If we dropped that condition we would have to work with L-groups with decoration. The full result can be found in [**?**].

Theorem 2.1. *Let G be a finitely presented group with $Wh(G) = e$. Then $L_n(G \times \mathbb{Z}) \cong L_n(G) \oplus L_{n-1}(G)$.*

Since $\pi_1(T^n) \cong \mathbb{Z}^n$ fulfils the condition of the theorem we inductively conclude:

Corollary 2.2.

$$L_n(\mathbb{Z}^n) \cong \bigoplus_{i=0}^{n} \binom{n}{i} L_{n-i}(e).$$

Before we at least can understand parts of the proof idea we need the concepts of the splitting obstruction α and of a splittable map.

Remember that the set of normal invariants $[X, G/TOP]$ is in 1:1 correspondence to the set of normal maps $\mathscr{N}(X)$ if X is a manifold. We write elements as triples $[M, \varphi, F]$ following the notation in [**?**], i.e.

$$\begin{array}{ccc} \varphi^*\nu & \longrightarrow & \nu \\ \downarrow & & \downarrow \\ M & \xrightarrow{\varphi} & X \end{array}$$

is a map of degree 1 and $F : TM \oplus \varphi^*\nu \to M \times \mathbb{R}^N$ is a stable trivialisation for some sufficiently large N. A short consideration shows that this data is equivalent to the one we used so far for elements in $\mathscr{N}(X)$. Under the above correspondence there is a surgery obstruction map on $[X, G/TOP]$. In [**?**] this map is called S.

We can now turn to the concept of a a map which is splittable.
Therefore we regard the following setting. Let X be a topological manifold of dimension $n > 5$ with a codimension k submanifold L. Let M be another manifold such that $\varphi : M \to X$ is a homotopy equivalence.

Definition 2.3. The map φ is *splittable* if there exists a $\psi \simeq \varphi$ such that $\mathrm{Im}\psi \pitchfork L$ and $\psi : (M, \psi^{-1}(L)) \to (X, L)$ is a homotopy equivalence.

Remark 2.4.
- Assume that the fundamental groups are $\pi_1(L) \cong G$ for some group G and $\pi_1(X) \cong G \times \mathbb{Z}$, where $\mathrm{codim}L = 1$. Farrell and Hsiang proved in [**?**] that a map is splittable if and only if the 'projection of the Whitehead torsion to the Grothendieck group of projective right modules over the integral group ring of G with an involution' vanishes.
- As mentioned before in our situation the whole Whitehead group vanishes so homotopy equivalences to a torus is always be splittable.
- In the last talk we saw an example of a homotopy which is not split: Regard the codimension 2 submanifold $\mathbb{C}P^4 \to \mathbb{C}P^5$ and the homotopy equivalence $f : \widetilde{\mathbb{C}}P^5 \to \mathbb{C}P^5$ then $f^{-1}(\mathbb{C}P^4) \simeq \mathbb{C}P^4 \sharp M$ for some Milnor manifold. In particular $\mathbb{C}P^4$ and the transversal preimage are not homotopy equivalent.

Again regard X and $L \subset X$ as above but with fixed codimension 1. Furthermore let $[M, \varphi, F] \in \mathscr{N}(X)$ be a normal map of degree one and assume that $\mathrm{Im}\varphi \pitchfork L$.

Definition 2.5. Denote by $N := \varphi^{-1}(L)$ the $(n-1)$-dimensional submanfold of X and by $\psi := \phi|_N$ the restriction of φ to this submanifold. We also get induced normal bundle data $\widetilde{F}$. Then we define the *splitting obstruction*

$$\begin{aligned} \alpha_L : \mathscr{N}(X) &\to L_{n-1}(G) \\ \alpha_L([M, \varphi, F]) &:= \sigma([N, \psi, \widetilde{F}]). \end{aligned}$$

Note that this definition strongly depends on exercise 5 to see that $(N, \psi, F|_N)$ is again a degree 1 normal map with a trivialisation $TN \oplus \nu(N \to \mathbb{R}^\infty)$ and then to get well-definedness, i.e. the independence of the representative for $[M, \varphi, F]$.
Now we assembled the necessary background to at least introduce one of the maps involved in the Shaneson splitting (2.1).

Proofidea:

To prove the theorem one constructs maps $\alpha : L_n(G \times \mathbb{Z}) \to L_{n-1}(G)$ and $\beta : L_n(G \times \mathbb{Z}) \to L_n(G)$.

We restrict to the definition of the first map which is rather easy after our considerations so far. As the name suggests we want to employ the map α_L for a cleverly picked L.
Fix $\eta \in L_n(\pi_1(X))$ with $\pi_1(X) \cong G \times \mathbb{Z}$ and $\mathrm{dim}X = n - 1$. Owing to the special form of our fundamental group and the assumtion that G is always finitely presentable X is of the form $K \times S^1$ with $\pi_1(K) \cong G$.
By Wall's realisation theorem there always exists a manifold W and a degree 1 map $\varphi_W : (W, \partial_+ W, \partial_- W) \to (K \times S^1 \times I, K \times S^1 \times \{0\}, K \times S^1 \times \{1\})$ such that $[W, \varphi, F] \in \mathscr{N}_n(K \times S^1 \times I)$ and $\sigma([W, \varphi, F]) = \eta$.
In the definition of the splitting obstruction α_L we now set $L := K \times I$ whence

$$\alpha(\eta) := \alpha_{K \times I}([W, \varphi, F]).$$

3. The surgery obstrcution map $\sigma : [T^n, G/TOP] \to L_n(\mathbb{Z}^n)$ is injective

We do exactly prove what the title claims namly

Lemma 3.1. *$\sigma : [T^n, G/TOP] \to L_n(\mathbb{Z}^n)$ is injective for all n.*

As remarked before the proof of Theorem 1.1 does not work in low dimensions. This Lemma however can be proven for all n by employing a little trick.

Proof. The proof is inductive. The induction start follows easily since $T^1 = S^1$ so $[T^1, G/TOP] = \pi_1(G/TOP) = 0$.
Therefore we assume that $n \geq 2$ and for $\xi \in [T^{n-1}, G/TOP]$ we know $\sigma(\xi) = 0 \Rightarrow \xi = 0$.
Fix any $T^{n-1} \overset{i}{\subset} T^n$, i denoting the inclusion. Regard the two projections on the first factor

$$\begin{aligned} \pi_n : T^n \times \mathbb{C}P^2 &\to T^n \\ \pi_{n-1} : T^{n-1} \times \mathbb{C}P^2 &\to T^{n-1}. \end{aligned}$$

This is the trick. Owing to periodicity of the L-groups the surgery obstructions of an element in $[T^n, G/TOP]$ pulled back by the projection has the same value in $L_n(\mathbb{Z}^n) \cong L_{n+4}(\mathbb{Z}^n)$ but it allows us to work in the 'nice' dimensions ≥ 5 where we have the surgery long exact sequence .
Fix some element $\xi \in [T^n, G/TOP]$ such that $\sigma(\xi) = 0 \in L_n(\mathbb{Z}^n)$.
This implies $\sigma(\pi_n \circ \xi) = 0 \in L_{n+4}(\mathbb{Z}^n)$. By the surgery exact sequence there exists $h : W \to T^n \times \mathbb{C}P^2$ a (simple) homotopy equivalence which is mapped to $\xi \circ \pi_n \in [T^n \times \mathbb{C}P^2, G/TOP]$ by η.
We have the submanifold $T^{n-1} \times \mathbb{C}P^2 \subset T^n \times \mathbb{C}P^2$ and, after making h transversal we can regard $N := h^{-1}(T^{n-1} \times \mathbb{C}P^2)$.
Since $Wh(\mathbb{Z}^n) = 0$ the homotopy equivalence h is splittable i.e. we can assume that $f := h|_N : N \to T^{n-1} \times \mathbb{C}P^2$ is a homotopy equivalence, too. From the correspondence $[T^n, G/TOP] \cong \mathscr{N}(T^n)$ and the definition of η we get

$$\eta(f) = \xi \circ \underbrace{\pi_n \circ (i \times 1_{\mathbb{C}P^2})}_{i \circ \pi_{n-1}}.$$

Since $\xi \circ i \circ \pi_{n-1}$ is in the image of η its surgery obstruction vanishes. By induction hypothesis that means that $\xi \circ i \circ \pi_{n-1} = 0 \in [T^{n-1} \times \mathbb{C}P^2, G/TOP]$.
Using the loop space property of $G/TOP = \Omega Y$ and the fact that $\Sigma T^n \simeq \vee \binom{n}{i} S^i$ we get

$$[\Sigma T^n, G/TOP] = [\vee_j S^i_j, Y] \Rightarrow [T^n, G/TOP] \cong \bigoplus \binom{n}{i} \pi_i(G/TOP).$$

Comparing with $[T^{n-1}, G/TOP]$ for all subtori we conclude that ξ is trivial on the $(n-1)$-skeleton. Hence we only need to consider the top cell. We have the collaps map $c : T^n \to S^n$ inducing the commutative diagram

$$\begin{array}{ccc} [S^n, G/TOP] & \xrightarrow{\sigma} & L_n(e) \\ \downarrow{\scriptstyle c^*} & & \downarrow \\ [T^n, G/TOP] & \xrightarrow{\sigma} & L_n(\mathbb{Z}^n) \end{array}$$

The upper obstruction map is an isomorphism by the Poincare conjecture. The right one is injective by the functoriality of L groups and since the composition $\{e\} \to \mathbb{Z}^n \to \{e\}$ is the identity. Therefore the induced composition on L-groups must also be the identity, i.e. $L_n(e) \to L_n(\mathbb{Z}^n)$ is injective.
By the commutativity we get that ξ also vanishes on the top cell and thereby on all of T^n. □

4. σ_1 SURJECTIVE

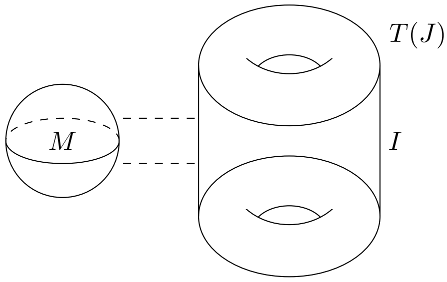
Let $\overline{n}$ denote $\{1, \dots, n\}$.

Lemma 4.1. *$\sigma_1 \colon [\Sigma T^n_+, \mathrm{G/TOP}] \to L_{n+1}(\mathbb{Z}^n)$ is surjective or, equivalently, $\partial \colon L_{n+1}(\mathbb{Z}^n) \to \mathscr{S}^{\mathrm{TOP}}(T^n)$ is trivial*

Definition 4.2. Let $J \subseteq K \subseteq \overline{n}, H = K \setminus J$ and

$$T(J) = \big\{(x_1, \dots, x_n) \mid x_i = \text{base point of } S^1 \text{ for } i \notin J\big\} \subset T^n$$

We construct an element $\xi(J, K) \in L_{|K|+1}(\mathbb{Z}^{|K|})$. Choose $(M, h, F) \in \mathscr{N}(D^{|J|+1})$ with M a closed manifold such that $\sigma(M, h, F)$ generates $L_{|J|+1}(e)$. From the previous talks we know that this is the Milnor or Kervaire manifold, depending on the dimension. Then we take the connected sum with $T(J) \times I$ using embeddings $D^{|J|+1} \to M$ and $D^{|J|+1} \to T \times (0, 1)$



and define $\xi(J, K) := \sigma(\eta)$ as the surgery obstruction of

$$\eta := ((T(J) \times I \amalg M) \times T(H), (\mathrm{id} \amalg h) \times \mathrm{id}, E \times D)$$
$$\in \mathscr{N}_{|K|+1}((T(J) \times I \amalg D^{|J|+1}) \times T(H)).$$

where D is the standard framing of $\tau_H \oplus \nu_H$, E a stable framing of $\tau_{T(J) \times I \amalg M} \oplus (\mathrm{id} \amalg h)^* \nu_{J \times I}$ and $\nu_{J \times I}$ and ν_H the stable normal bundles of $T(J) \times I$ and $T(H)$. We write $\xi(J) := \xi(J, \overline{n})$

Lemma 4.3. *$\xi(J) \in \ker \partial$ for all $J \subseteq \overline{n}$*

Proof. Since $T(J) \times I \times T(H)$ is a manifold η considered as an element of $\mathscr{N}(T(J) \times I \times T(H) \operatorname{rel} \partial)$ defines an element in $[\Sigma T^n_+, G/TOP]$ and $\xi(J)$ agrees with its image under σ_1. The statement follows from the exactness of the surgery sequence. $\square$

Proof of Theorem 4.1. By Lemma 4.3 we have

$$G := \left\{ \sum_{\emptyset \neq J \subseteq \overline{n}} b_J \xi(J) \mid b_J \in \mathbb{Z} \right\} \subset \ker \partial$$

Define
$$p \colon G \to \bigoplus_{\emptyset \neq H \subseteq \overline{n}} L_{n+1-|H|}(e)$$
by
$$\xi(J) \mapsto p_H \circ \alpha(H)(\xi(J))$$
where $p_H \colon L_{|H|+1}(\pi_1(T(H))) \to L_{|H|+1}(e)$ is the map induced by $\pi_1 T(H) \to e$ and for $\alpha(H)$ choose a sequence $H = H_0 \subset H_1 \ldots \subset H_k = \overline{n}$ with $|H|_i = |H|_{i-1} + 1$ and define

$$\alpha(H) := \alpha(H_0, H_1) \circ \dots \circ \alpha(H_{k-1}, H_k),$$

where $\alpha(H_i, H_{i+1})$ denotes the splitting obstruction of the submanifold H_i in H_{i+1}. By the Shaneson splitting we know that $L_{n+1}(\mathbb{Z}^n)$ is isomorphic to $\bigoplus_{\emptyset \neq H \subseteq \overline{n}} L_{n+1-|H|}(e)$

and finitely generated abelian. Hence it is sufficient to prove that p is surjective which is a consequence of the following lemma. ☐

Lemma 4.4.

$$p_H \circ \alpha(H)(\xi(J)) = \begin{cases} p_J(\xi(J,J)) = 1 & \textit{if } J = H \\ 0 & \textit{if } J \not\subseteq H \\ (p_H(\xi(J,H)) = 0 & \textit{if } J \subsetneq H) \end{cases}$$

Proof. The case $J = H$ is apparent by the definition of $\xi(J)$. If $J \not\subseteq H$ we can attach M to $T(J)$ in $\xi(J)$ in such a way that it doesn't meet $T(H)$ and so $\xi(J)$ becomes trivial if we restrict via $\alpha(H)$ to the $T(H)$ part. The case $J \subsetneq H$ is not neccessary for the proof of Lemma 4.1 but can be proven by using the product formula for simply connected surgery. ☐

5. Exercises

Exercise 1. Proof that the suspended torus has the homotopy type of a wedge of spheres:

$$\Sigma T^n \simeq \bigvee_{i=0}^{n} \binom{n}{i} S^{i+1}$$

Solution. Use the standard cell structure of T^n to define a map

$$c\colon T^n \to \bigvee_{0 \le i \le n} \binom{n}{i} S^i$$

which induces isomorphims on homology

$$c_k\colon H_k(T^n;\mathbb{Z}) = \mathbb{Z}^{\binom{n}{k}} \xrightarrow{\cong} H_k\left(\bigvee_{i=0}^{\binom{n}{k}} S^k;\mathbb{Z}\right).$$

Suspension leads to an isomorphism on fundamental groups and hence by Whiteheads theorem to a homotopy equivalence.

Exercise 2. Let $(f,b)\colon (M,\nu_M) \to (X,\xi)$ be a degree one normal map. For simplicity, assume that M and X are closed oriented Cat-manifolds of dimension n. Suppose that $Y \subset X$ is a codimension k oriented submanifold X with normal bundle $\nu_{Y\subset X}$ and that that f is transverse to Y. Prove the following:

(1) There is a canonical degree one normal map

$$(f|_N, b')\colon (N,\nu) \to (Y, \xi|_Y \oplus \nu_{Y\subset X}).$$

(2) This defines a well-defined map $\pitchfork_Y\colon \mathcal{N}(X,\xi) \to \mathcal{N}(Y,\xi|_Y \oplus \nu_{Y\subset X})$

Solution. Set $N = f^{-1}(Y)$, $b' = b|N$ and $\nu = b'^*(\xi|Y \oplus \nu_{Y\subset X}) = (\nu_M|N \oplus \nu_{N\subset M})$.

The induced map on the Thom spaces $\mathrm{Th}(b) : \mathrm{Th}(\nu_M) \to \mathrm{Th}(\nu_X)$ sends the Thom class u_M of ν_M to the Thom class u_X of ν_X. The same holds for $Th(b')$ and the Thom classes $u_\nu \in \mathrm{Th}(b'^*(\xi|Y \oplus \nu_{Y\subset X}))$ and $u_Y \in \mathrm{Th}(\xi|Y \oplus \nu_{Y\subset X})$. We get the following commutativ diagramm.

$$\begin{array}{ccc} \mathrm{Th}(b'^*(\xi|Y \oplus \nu_{Y\subset X})) & \xrightarrow{\mathrm{Th}(b')} & \mathrm{Th}(\xi|Y \oplus \nu_{Y\subset X}) \\ & u_\nu \mapsto u_Y & \\ \downarrow & & \downarrow \\ \mathrm{Th}(b'^*\xi|Y) \wedge \mathrm{Th}(b'^*\nu_{Y\subset X}) & \longrightarrow & \mathrm{Th}(\xi|Y) \wedge \mathrm{Th}(\nu_{Y\subset X}) \\ & \widetilde{u}_N \wedge u_{N\subset M} \mapsto \widetilde{u}_Y \wedge u_{Y\subset X} & \end{array}$$

Let r be the dimension of ξ. The Thom isomorphism and Poincaré duality give

$$\begin{array}{ccccc}
H^{r+k}(\mathrm{Th}(\nu)) & & \xrightarrow{\mathrm{Th}(b')} & & H^{r+k}(\mathrm{Th}(\xi|Y \oplus \nu_{Y\subset X})) \\
\downarrow & u_\nu & \longleftarrow & u_Y & \downarrow \\
& \vdots & & \vdots & \\
H^0(N) & & \xleftarrow{(f|_N)^*} & & H^0(Y) \\
\downarrow & 1 & \longleftarrow & 1 & \downarrow \\
& \vdots & & \vdots & \\
H_k(N) & & \xrightarrow{(f|_N)_*} & & H_k(Y) \\
& [N] & \longmapsto & [Y] &
\end{array}$$