1. Manifold Atlas : Regensburg Surgery Blockseminar 2012
Exotic spheres (Sebastian Goette)

1.1. **The surgery sequence for spheres.** Recall the long exact surgery sequence for spheres from the previous talk,

$$\cdots \longrightarrow L_{n+1}(\mathbb{Z}) \longrightarrow \mathscr{S}_n(S^n) \xrightarrow{\eta} \mathscr{N}_n(S^n) \longrightarrow L_n(\mathbb{Z}) \longrightarrow \cdots \longrightarrow L_5(\mathbb{Z}) ,$$

with

$$\mathscr{S}_n(S^n) = \{[f\colon \Sigma \to S^n] \mid \dots\} /\text{h-cob}$$

From the solution of the Poincaré conjecture: All Σ above are homeomorphic to S^n.

Theorem 1.1 (Hopf)**.** *All $f\colon \Sigma \to S_n$ of degree one are homotopic.*

This allows us to forget the map f in the definition of elements of the structure set.

Corollary 1.2 (from h-cobordism theorem)**.** *If $n \neq 4$, then*

$$\mathscr{S}^n(S^n) = \{ \text{ oriented diffeo types on } S^n\} =: \Theta^n .$$

With connected sum, Θ^n becomes a group, the group of *exotic spheres* in dimension n. The inverse of an exotic sphere is the same manifold with the opposite orientation.

Definition 1.3 ([4, Definition 6.8])**.** An *almost framed n-manifold* $(M, x_0, \bar{u})$ is an n-manifold M with a strong vector bundle isomorphism

$$\bar{u}\colon T(M \setminus \{x_0\}) \oplus \underline{\mathbb{R}}^a \cong \underline{\mathbb{R}}^{n+a}$$

an *almost framed bordism* $(W, \gamma, \bar{v})$ between $(M_i, x_i, \bar{u}_i)$ for $i = 0, 1$ consists of a manifold W with $\partial W = M_0 \amalg M_1$, a path $\gamma\colon [0,1] \to W$ hitting $\partial_i W$ transversally in x_i at $t = i$, and a stable framing $\bar{v}$ of $T(W \setminus \operatorname{im}\gamma)$ that coincides with $\bar{u}_i$ on M_i. Let the *almost framed bordism group* Ω_n^{alm} be the set of classes with disjoint union as addition.

If we demand that the framings extend over all of M or W, we analogously get the *framed bordism groups* Ω_n^{fr}.

Proposition 1.4 ([4, Lemma 6.9])**.** *There is a natural isomorphism*

$$\mathscr{N}_n(S^n) \cong \Omega_n^{\text{alm}}$$

Proof. Let $f\colon (M, TM \oplus \underline{\mathbb{R}}^a) \to (S^n, \xi)$ be a smooth tangential normal map. Fix $s \in S^n$. After homotopy s is a regular value, and $f^{-1}(s) = \{x_0\}$ (because $\deg f = 1$). The bundle data gives a stable almost framing because $\xi|_{S^n \setminus \{s\}}$ is trivial. Analogously, bordisms of normal maps can be turned into almost framed bordisms.

For the inverse construction, let $\bar{u}\colon T(M \setminus \{x_0\}) \oplus \underline{\mathbb{R}}^a \cong \underline{\mathbb{R}}^{n+a}$ be an almost framing of M. Assume that M carries a Riemannian metric and that ε is very small. By contracting $M \setminus D_\varepsilon(x_0)$ to a point, we construct a map $f\colon M \to S^n$ with $x_0 \mapsto s$. The almost framing $\bar{u}$ induces a fibrewise isomorphism $\bar{f}\colon TM \oplus \underline{\mathbb{R}}^a \to \xi$ that is unique up to isomorphism. Similarly, an almost framed bordism gives rise to a bordism of normal maps. □

The long exact sequence from above becomes

$$\dots \longrightarrow \Omega_{n+1}^{\text{alm}} \longrightarrow L_{n+1}(\mathbb{Z}) \longrightarrow \Theta^n \longrightarrow \Omega_n^{\text{alm}} \longrightarrow L_n(\mathbb{Z}) \longrightarrow \dots$$

Definition 1.5. If an exotic sphere bounds a framed manifold we say $\Sigma \in bP^{n+1}$ (*b*oundary of something *p*arellelizable in dimension $n + 1$).

Note that $bP^{n+1} \subset \Theta_n$ is a subgroup under connected sum.

Proposition 1.6 ([4, Lemma 6.6])**.** *We have*

$$bP^{n+1} = \ker\bigl(\eta\colon \Theta^n \to \Omega_n^{alm}\bigr)\ .$$

Proof. If $\Sigma \in \ker\eta \subset \Theta^n$, then there exists a bordism $(F,\overline{F})\colon (W, TW \oplus \underline{\mathbb{R}}^b) \to (S^n,\eta)$ from the neutral element

$$(\mathrm{id},\bar{f}_0)\colon (S^n, TS^n \oplus \underline{\mathbb{R}}) \to (S^n, \underline{\mathbb{R}}^{n+1})$$

to the element $(f,\bar{f})\colon (\Sigma, T\Sigma \oplus \underline{\mathbb{R}}^a) \to (S^n,\xi)$ of $\mathscr{N}_n(S^n)$. The isomorphism $\bar{v}_0\colon \eta \cong \underline{\mathbb{R}}^{n+1+b}$ trivialises η, so we may assume that $\eta = \underline{\mathbb{R}}^{n+1+b}$. Consider the manifold

$$N = W \cup_{\partial_0 W} D^{n+1}$$

and extend $\overline{F}$ to $TN \oplus \underline{\mathbb{R}}^b \to \underline{\mathbb{R}}^{n+1+b}$. Then $\overline{F}$ becomes a stable framing of N, so $\Sigma = \partial N \in bP^{n+1}$.

Conversely, let $\Sigma = \partial N \in bP^{n+1}$, and let $f\colon \Sigma \to S^n$ be a smooth map of degree one. Then f can be extended to a smooth map $F\colon N \to D^{n+1}$ of degree one, and we may assume that 0 is a regular value of F with $F^{-1}(0) = \{x\}$ a single point. For small $\varepsilon > 0$, the preimage $F^{-1}(D_\varepsilon(0))$ is again a disk. Hence we obtain a bordism $W = N \setminus F^{-1}(D_\varepsilon(0))$ with $\partial_0 W = S^N$, $\partial_1 W = \partial N = M$, and with a map

$$F\colon W \to D^{n+1} \setminus D_\varepsilon(0) \longrightarrow S^n\ .$$

The stable framing of N gives rise to a map $\overline{F}\colon TN \oplus \underline{\mathbb{R}}^b \to \underline{\mathbb{R}}^{n+1+b}$, so we have constructed a normal bordism. This shows that $bP^{n+1} \subset \ker\eta$. $\square$

The long exact surgery sequence now splits into tractable pieces,

(1) $$0 \longrightarrow \Theta^{4k} \xrightarrow{\eta} \Omega^{\mathrm{alm}}_{4k} \xrightarrow{\frac{\mathrm{sgn}}{8}} \mathbb{Z} \xrightarrow{\partial} bP^{4k} \longrightarrow 0 \qquad \text{for } k \geq 2\ ,$$

(2) $$0 \longrightarrow \Theta^{4k-2} \xrightarrow{\eta} \Omega^{\mathrm{alm}}_{4k-2} \xrightarrow{\mathrm{Arf}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\partial} bP^{4k-2} \longrightarrow 0 \qquad \text{for } k \geq 2\ , \text{ and}$$

(3) $$0 \longrightarrow bP^{2j} \longrightarrow \Theta^{2j-1} \xrightarrow{\eta} \Omega^{\mathrm{alm}}_{2j-1} \longrightarrow 0 \qquad \text{for } j \geq 3\ .$$

1.2. **Another exact sequence.** Recall the framed bordism groups Ω^{fr}_n of Definition 1.3. The Pontrijagin-Thom construction gives a natural isomorphism

$$\Omega^{\mathrm{fr}}_n \xrightarrow{\cong} \pi^s_n = \operatorname{colim}_{b\to\infty} \pi_{n+b}(S^b)\ .$$

Given a map $\alpha\colon S^n \to O(b)$, we get a family of rotations of S^{b-1}. Evaluation gives a map $S^n \times S^{b-1} \to S^{b-1}$. This induces a map

$$f_\alpha\colon S^{n+b} \cong S^n * S^{b-1} \longrightarrow \Sigma S^{b-1} \cong S^b\ ,$$

where $*$ denotes the join and Σ denotes unreduced suspension. This is one way to define the *J-homomorphism*

$$J\colon \pi_n(O) \longrightarrow \pi^s_n \cong \Omega^{\mathrm{fr}}_n \qquad \text{with} \qquad J[\alpha] = [f_\alpha]\ .$$

If M is almost framed, we can represent $TM \oplus \underline{\mathbb{R}}^b$ as the pullback of a bundle $\xi \to S^{4k}$ along the collapse map

$$f\colon M \longrightarrow M/(M \setminus D_\varepsilon(p)) \xrightarrow{\cong} S^{4k}\ .$$

Stable vector bundles over S^n are classified by $\pi_n(\mathrm{BO})$, so we have a map $\partial\colon \Omega^{\mathrm{fr}}_n \to \pi_n(\mathrm{BO})$.

Finally, there is a natural forgetful map $\Omega^{\mathrm{fr}}_n \to \Omega^{\mathrm{alm}}_n$.

Lemma 1.7. *There is a long exact sequence*

(4) $$\dots \longrightarrow \pi_{n+1}(\mathrm{BO}) \xrightarrow{J} \underbrace{\Omega^{\mathrm{fr}}_n}_{=\pi^s_n} \longrightarrow \Omega^{\mathrm{alm}}_n \xrightarrow{\partial} \pi_n(\mathrm{BO}) \longrightarrow \dots$$

Proof. Exercise. $\square$

1.3. **The signature of almost framed bordism classes.** Hirzebruch's signature theorem says that

$$\operatorname{sgn}(M) = L(TM)[M] .$$

Let $[M] \in \Omega_{4k}^{\mathrm{alm}}$ and consider the collapse map $(f, \bar{f}) \colon (M, TM \oplus \mathbb{R}^b) \to (S^{4k}, \xi)$ in the definition of the map ∂ in (4). Because Pontrijagin classes are natural and f has degree one, we conclude that

$$\operatorname{sgn}(M) = L(\xi)[S^n] .$$

Now, the computation of the possible values of $\operatorname{sign}(M)$ proceeds in two steps.

(1) Identify the image of $\Omega_{4k}^{\mathrm{alm}}$ in $\widetilde{KO}_0(S^{4k}) \cong \pi_{4k-1}(O)$.
(2) Determine $L(\xi)[S^{4k}]$ for these vector bundles.

We use Lemma 1.7 to determine $\operatorname{im} \partial = \ker J$. The groups $\pi_n(O) = \pi_{n+1}(\mathrm{BO})$ are periodic and satisfy

$$\pi_n(O) = \begin{cases} \mathbb{Z} & \text{for } n \equiv 3,\, 7 \bmod 8, \\ \mathbb{Z}/2\mathbb{Z} & \text{for } n \equiv 0,\, 1 \bmod 8, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Regard the *Bernoulli numbers* $B_n \in \mathbb{Q}$ with

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{B_n}{(2n)!} t^{2n} .$$

The first values are

n	1	2	3	4	5	6	7	8
B_n	$\frac{1}{6}$	$\frac{1}{30}$	$\frac{1}{42}$	$\frac{1}{30}$	$\frac{5}{66}$	$\frac{691}{2730}$	$\frac{7}{6}$	$\frac{3617}{510}$

Theorem 1.8 (Adams, [1, Theorems 1.1, 1.3, 1.5])**.** *If $n \neq 3 \bmod 4$, then the J-homomorphism $J \colon \pi_n(O) \to \pi_n^s$ is injective. Otherwise $\# \ker J = \operatorname{denom} \frac{B_k}{4k}$.*

We come to step (2). Computations with characteristic classes and a little index theory can be used to show the following.

Theorem 1.9. *A generator ξ of $\widetilde{\mathrm{KO}}_0(S^{4k}) \cong \pi_{4k}(\mathrm{BO}) \cong \pi_{4k-1}(O)$ satisfies*

$$L(\xi)[S^{4k}] = a_k \frac{2^{2k}(2^{2k} - 1) B_k}{2k}$$

where $a_k = 1$ for k even and $a_k = 2$ for k odd.

Combine these two theorems to get

Theorem 1.10.

$$\# bP^{4k} = a_k \cdot 2^{2k-2}(2^{2k-1} - 1) \operatorname{num} \frac{B_k}{4k}$$

and bP^{4k} is cyclic.

The first values are

k	2	3	4	5	6	7
$\# bP^{4k}$	$2^2 \cdot 7$	$2^5 \cdot 31$	$2^6 \cdot 127$	$2^9 \cdot 7 \cdot 73$	$2^{10} \cdot 23 \cdot 89 \cdot 691$	$2^{13} \cdot 8191$

The value for $k = 1$ is meaningless—there are no exotic spheres in dimension 3.

Proof. We have seen that the map $\sigma = \frac{\mathrm{sign}}{8}$ factors over $\pi_{4k}(\mathrm{BO}) \cong \widetilde{\mathrm{KO}}_0(S^{4k})$. From (1),

$$\begin{array}{ccccccc} \Omega_4k^{\mathrm{alm}} & \xrightarrow{\frac{\mathrm{sign}}{8}} & \mathbb{Z} & \longrightarrow & bP^{4k} & \longrightarrow & 0 \\ & \overset{\partial}{\searrow} & \big\uparrow{\scriptstyle \frac{1}{8}L(\,\cdot\,)[S^{4k}]} & & & & \\ & & \pi_{4k}(\mathrm{BO}) & & & & \end{array}$$

Theorem 1.8 computes $\operatorname{im}\partial = \ker J$ and Theorem 1.9 computes the L-number. □

Remark 1.11. The Eells-Kuiper invariant μ detects all elements of bP^7 and bP^{11}, see [2]. For $\Sigma \in bP^{4k-1}$, one either chooses N parallelisable with $\partial N = \Sigma$ and takes a linear combination of $\mathrm{sign}(N)$ and certain characteristic numbers of N. Or one takes a linear combination of certain η-invariants and Cheeger-Chern-Simons correction terms directly on Σ. For higher k, Stolz's s-invariant also distinguishes all elements of bP^{4k-1}. It can be defined and computed in a similar manner [5].

1.4. **The Kervaire-Invariant-One-Problem.** Because $\pi_{4k-2}(\mathrm{BO}) = 0$, the forgetful map in (4) is surjective for $n = 4k-2$.

Theorem 1.12 (Kervaire, Milnor, ..., Hill, Hopkins, Ravenell [3])**.** *The homomorphism*

$$\mathrm{Arf}\colon \Omega^{\mathrm{fr}}_{4k-2} \cong \pi^s_{4k-2} \longrightarrow \mathbb{Z}_2$$

is surjective for $4k-2 \in \{2, 6, 14, 30, 62\}$*, trivial for* $4k-2 \notin \{2, 6, 14, 30, 62, 126\}$*, and not yet known for* $4k-2 = 126$*.*

Note: exception are of the form $2^j - 2$.

Theorem 1.13. *The group* bP^{4k-2} *are either trivial or* $\mathbb{Z}_2$ *with*

$$bP^{4k-2} \cong \begin{cases} 0 & \text{if } 4k-2 \in \{6, 14, 30, 62\} \\ \mathbb{Z}_2 & \text{if } 4k-2 \, \{6, 14, 30, 62, 126\} \end{cases}$$

Proof. Use sequence (2). □

1.5. **The Cokernel of the J-homomorphism.** We now use the sequence (3) and the left hand sides of the sequences (1) and (2) to determine the remaining groups Θ_n/bP^{n+1}. For even n, $bP^{n+1} = 0$, so $\Theta_n = \Theta_n/bP^{n+1}$.

By Theorem 1.8, the J-homomorphism $J\colon \pi_n(O) \to \Omega^{\mathrm{fr}}_n \cong \pi^s_n$ is injective except if $n \equiv 3 \bmod 4$. Hence if $n \not\equiv 0 \bmod 4$, sequence (4) becomes

$$\pi_n(O) \xrightarrow{J_n} \Omega^{\mathrm{fr}}_n \longrightarrow \Omega^{\mathrm{alm}}_n \xrightarrow{\partial} 0 = \ker\bigl(J_{n-1}\colon \pi_{n-1}(O) \to \pi^s_{n-1}\bigr) ,$$

in particular,

$$\Omega^{\mathrm{alm}}_n \cong \operatorname{coker}\bigl(J_n\colon \pi_n(O) \to \pi^s_n\bigr) .$$

If $n = 4k$, we know that

$$\ker\Bigl(\frac{\mathrm{sign}}{8}\colon \Omega^{\mathrm{alm}}_{4k} \to \mathbb{Z}\Bigr) = \ker\bigl(\partial\colon \Omega^{\mathrm{alm}}_{4k} \to \pi_{4k-1}(O)\bigr)$$

because $\frac{\mathrm{sign}}{8}$ factors through the map $\frac{1}{8}L(\,\cdot\,)[S^{4k}]\colon \pi_{4k}(BO) \cong \pi_{4k-1}(O) \to \mathbb{Z}$, which is injective by Theorem 1.9. As above, sequence (4) implies that

$$\ker\Bigl(\frac{\mathrm{sign}}{8}\colon \Omega^{\mathrm{alm}}_{4k} \to \mathbb{Z}\Bigr) \cong \operatorname{coker}\bigl(J_n\colon \pi_n(O) \to \pi^s_n\bigr) .$$

Because the Arf invariant is not completely understood, we finally get the following picture.

Theorem 1.14 ([4, Theorem 6.46]). *If $n \in \{6, 14, 30, 62\}$, there is a short exact sequence*

$$0 \longrightarrow \Theta_n/bP^{n+1} \longrightarrow \operatorname{coker} J_n \longrightarrow \mathbb{Z}_2 \longrightarrow 0 \,. \tag{1}$$

If $n \notin \{6, 14, 30, 62, 126\}$, then

$$\Theta_n/bP^{n+1} \cong \operatorname{coker} J_n \,. \tag{2}$$

For $n = 126$, either (1) or (2) holds.

Corollary 1.15. *All exotic spheres are stably parallelisable. Unless $n \in \{6, 14, 30, 62, 126\}$, all framed bordism classes are realised by exotic spheres.*

It follows from Theorem 1.8 that $\operatorname{coker} J_n$ is closely related to the mysterious stable homotopy groups of spheres. The following table gives the orders of the first few groups of exotic spheres. We do not go into their group structures here.

n	5	6	7	8	9	10	11	12	13	14	15	16	17
$\#bP^{n+1}$	**1**	1	28	1	2	1	992	1	**1**	1	8128	1	2
$\#\pi_n^s$	1	2	240	4	8	6	504	1	3	4	960	4	16
$\#\operatorname{coker} J_n$	1	2	1	**2**	**4**	6	1	1	3	4	2	**2**	**8**
$\#(\Theta_n/bP^{n+1})$	1	**1**	1	2	4	6	1	1	3	**2**	2	2	8
$\#\Theta_n$	1	1	28	2	8	6	992	1	3	2	16256	2	16

Fat numbers either represent exceptions due to the Kervaire invariant one problem, see Theorems 1.13 and 1.14, or they are due to $\pi_{8k+i}(\mathrm{BO}) = \mathbb{Z}_2$ for $i \in \{1, 2\}$.

References

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