Manifold Atlas: supplementary material for the page **Complex bordism**

Lemma. *We have*

$$s_{i+j-1}[H_{ij}] = \begin{cases} j, & \text{if } i=0, \text{ i.e. } H_{ij}=\mathbb{C}P^{j-1};\\ 2, & \text{if } i=j=1;\\ 0, & \text{if } i=1,\ j>1;\\ -\binom{i+j}{i}, & \text{if } i>1. \end{cases}$$

Proof. Let $i=0$. Since the stably complex structure on $H_{0j}=\mathbb{C}P^{j-1}$ is determined by the isomorphism $\mathcal{T}(\mathbb{C}P^{j-1})\oplus\mathbb{C}\cong\bar\eta\oplus\ldots\oplus\bar\eta$ (j summands) and $x=c_1(\bar\eta)$, we have

$$s_{j-1}[\mathbb{C}P^{j-1}] = jx^{j-1}\langle\mathbb{C}P^{j-1}\rangle = j.$$

Now let $i>0$. Then

$$s_{i+j-1}(\mathcal{T}(\mathbb{C}P^i\times\mathbb{C}P^j)) = (i+1)x^{i+j-1}+(j+1)y^{i+j-1} = \begin{cases} 2x^j+(j+1)y^j, & \text{if } i=1;\\ 0, & \text{if } i>1. \end{cases}$$

Denote by ν the normal bundle of the embedding $\iota\colon H_{ij}\to\mathbb{C}P^i\times\mathbb{C}P^j$. Then

$$\mathcal{T}(H_{ij})\oplus\nu = \iota^*(\mathcal{T}(\mathbb{C}P^i\times\mathbb{C}P^j)).$$

Since $c_1(\nu)=\iota^*(x+y)$, we obtain $s_{i+j-1}(\nu)=\iota^*(x+y)^{i+j-1}$.

Assume $i=1$. Then by the previous Proposition,

$$\begin{aligned} s_j[H_{1j}] = s_j(\mathcal{T}(H_{1j}))\langle H_{1j}\rangle &= \iota^*(2x^j+(j+1)y^j-(x+y)^j)\langle H_{1j}\rangle\\ &= (2x^j+(j+1)y^j-(x+y)^j)(x+y)\langle\mathbb{C}P^1\times\mathbb{C}P^j\rangle\\ &= \begin{cases} 2, & \text{if } j=1;\\ 0, & \text{if } j>1. \end{cases} \end{aligned}$$

Assume now that $i>1$. Then $s_{i+j-1}(\mathcal{T}(\mathbb{C}P^i\times\mathbb{C}P^j))=0$, and by the previous Proposition,

$$s_{i+j-1}[H_{ij}] = -s_{i+j-1}(\nu)\langle H_{ij}\rangle = -\iota^*(x+y)^{i+j-1}\langle H_{ij}\rangle = -(x+y)^{i+j}\langle\mathbb{C}P^i\times\mathbb{C}P^j\rangle = -\binom{i+j}{i},$$

which finishes the proof of the Lemma. □

Taras Panov
Department of Geometry and Topology
Faculty of Mathematics and Mechanics
Moscow State University, Leninskie Gory
119991 Moscow RUSSIA

E-mail address: tpanov@higeom.math.msu.su