1. Manifold Atlas : Regensburg Surgery Blockseminar 2012
Cat surgery: A brief overview (Diarmuid Crowley)

Example 1.1 ([C]). There exists a short exact sequence of pointed sets

$$bP_8 \cong \mathbb{Z}/28 \longrightarrow \mathscr{S}^{\mathrm{Diff}}(S^3 \times S^4) \xrightarrow{\eta} \pi_4(G/O) \longrightarrow L_7(\mathbb{Z}) = 0.$$

If we identify $\pi_4(G/O) = \mathbb{Z}$ then the action of bP_8 is described as follows: bP_8 acts freely on $\eta^{-1}(v)$ if 7 divides $v \in \mathbb{Z}$ and $(bP_8 \cdot [v]) \equiv \mathbb{Z}/4$ if 7 is prime to v. *In particular, it is not possible to make η a group homomorphism.* There are fibre homotopy equivalences

$$S^3 \tilde{\times}_{12v\gamma} S^4 \xrightarrow{f_v} S^3 \times S^4.$$

where $\gamma \in \pi_3(SO_3) \subset \pi_3(SO_4)$ is a generator. In Example 1.3 below we show that $\eta(f_v) = v \in \pi_4(G/O)$. If follows that all the elements of of $\mathscr{S}(S^3 \times S^4)$ are of the form

$$\Sigma\sharp(S^3 \tilde{\times}_{12v\gamma} S^4) \xrightarrow{f_v} S^3 \times S^4$$

where $\Sigma \in bP_8$ and we regard $\Sigma\sharp(S^3 \tilde{\times}_{12v\gamma} S^4) := M_{v,\Sigma}$ as the same topological space as $(S^3 \tilde{\times}_{12n\gamma} S^4)$ in order to define f_v. Hence we may write elements of $\mathscr{S}(S^3 \times S^4)$ as $[\Sigma, f_v]$. In which case [C, Theorem 2.2] gives

$$M_{v_0,\Sigma_0} \cong_{\mathrm{diffeo}} M_{v_1,\Sigma_1} \Leftrightarrow v_0 = \pm v_1 \text{ and } \Sigma_0 - \Sigma_1 \in 2|v_0| \cdot bP_8.$$

Splitting invariants of normal maps. General setting: Let $Y^{n-k} \subseteq X^n$ be a codimension k submanifold and $(f, \overline{f})$ a degree one normal map with $f\colon M \to X$. After a small homotopy we assume that f is transverse to Y.

$$\begin{array}{ccc}
\nu_M & \longrightarrow & E \\
\downarrow{\scriptstyle \overline{f}} & & \downarrow \\
M & \xrightarrow{f} & X \\
\uparrow{\scriptstyle \subseteq} & & \uparrow{\scriptstyle \subseteq} \\
N := f^{-1}(Y) & \xrightarrow{f|_N} & Y
\end{array}$$

There exists canonical bundle data and a degree one normal map

$$\begin{array}{ccc}
\nu_n & \xrightarrow{\overline{f|_N}} & E|_Y \oplus \nu_{Y\subseteq X} \\
\downarrow & & \downarrow \\
N & \xrightarrow{f|_N} & Y
\end{array}$$

Lemma 1.2. *The normal bordism class of $(\overline{f|_N}, f|_N)$ depends only on the normal bordism class $[\overline{f}, f] \in \mathscr{N}(X)$. Assuming for simplicity that X and Y are orientable we obtain a well-defined map:*

$$\pitchfork_Y\colon \mathscr{N}(X) \to \mathscr{N}(Y), \quad [\overline{f}, f] \mapsto [\overline{f|_N}, f|_N]$$

We may then take the following composition

$$\mathscr{N}(X) \xrightarrow{\pitchfork_Y} \mathscr{N}(Y) \xrightarrow{\sigma_Y} L_{n-k}(\pi_1 Y, w_Y) \to L_{n-k}(\mathbb{Z})$$

where the last map is given since the L-groups are functorial for unital ring maps respecting the involution. In particular

$$(\mathbb{Z}\pi) \rightleftarrows \mathbb{Z}$$

$$g \longmapsto 1$$

is a split ring homomorphism.

Proof. Exercise. ☐

Example 1.3. Let $s\colon S^4 \to S^3 \mathbin{\tilde{\times}}_{12v\gamma} S^4$ be a section and identify $S^4 = s(S^4)$:

$$\begin{array}{ccc} S^3 \mathbin{\tilde{\times}}_{12v\gamma} S^4 & \xleftarrow{f_v^{-1}} & S^3 \times S^4 \\ {\scriptstyle s}\uparrow & & \uparrow{\scriptstyle i} \\ S^4 & \xleftarrow{f_v^{-1}|_N} & N \end{array}$$

Take $N := (f_v^{-1})^{-1}(S^4)$. Then we have the following bundle identifications:

$$\nu_N = \nu_i \oplus \nu_{S^3 \times S^4}$$

$$\nu_N = (f_v^{-1})^*(\nu_{S^4 \hookrightarrow S^3 \tilde{\times}_{12v\gamma} S^4}) \oplus \nu_{S^3 \times S^4} = (f_v^{-1})^* 24vS\gamma$$

Where $S\colon \pi_3(SO_3) \to \pi_3(SO)$ is the stabilisation map. From ν_N we can use the Signature Theorem to compute the signature of N and one finds that $\sigma(N) = 16v$. From this we conclude that the following composition

$$\mathcal{N}(S^3 \mathbin{\tilde{\times}}_{12v\gamma} S^4) \xrightarrow{\pitchfork_{S^4}} \mathcal{N}(S^4) \xrightarrow{\sigma} L_4(\mathbb{Z})$$

maps $[\overline{f}_v, f_v]$ to $\pm 2v \in L_4(\mathbb{Z})$. Applying Rohlin's Theorem, the homomorphism $\pi_4(G/O) \to L_4(\mathbb{Z})$ is seen to be multiplication by 2 and hence $[\overline{f}_v, f_v]$ maps to $v \in \mathbb{Z} = \pi_4(G/O)$.

Remark 1.4 (Invariants of structures/normal invariants). Consider

$$[f\colon M \to X] \in \mathcal{S}(X) \to \mathcal{N}(X).$$

There exist absolute invariants e.g. $\sigma_{4k}(M \to X) = \pm(\sigma_M - \sigma_X)$.

But there are invariants where the map f plays a role: Consider the following degree one normal map (bundle data supressed)

$$g\colon S^6 \times S^6 \to (S^6 \times S^6) \vee S^{12} \to S^6 \times S^6$$

which is a so-called pinch map [M-T-W] on a map $f\colon S^{12} \to S^6 = S^6 \times \mathrm{pt} \subset S^6 \times S^6$ and where we choose f to represent the generator of the stable 6-stem:

$$[f\colon S^{12} \to S^6] \in \pi_6^S \xrightarrow{\cong} \mathbb{Z}/2$$

We may take the transverse inverse image of g along $S^6 \times \mathrm{pt}$ and check, using [M-T-W] that $\sigma_6 \circ \pitchfork_{S^6}(\overline{g}, g) = 1 \in L_6(\mathbb{Z}) \cong \mathbb{Z}/2$. On the other hand we clearly have $\sigma_6 \circ \pitchfork_{S^6}(\overline{\mathrm{id}}, \mathrm{id}) = 0 \in L_6(\mathbb{Z})$. Now both the normal maps $(\overline{g}, g)$ and $(\overline{\mathrm{id}}, \mathrm{id})$ have the same source and target, so we see that the map may plan an important role in the invariants of normal maps.

***PL* manifolds and *TOP* manifolds.** We briefly describe how to study the relations between smooth manifolds, piecewise linear manifolds and topological manifolds. For more information see [H-M, K-S] and references for the statements which follow. We have the following schema of forgetful functors:

$$\text{Diff “}\!\to\!\text{” PL} \to \text{TOP}$$

The topological category, TOP, consists of topological manifolds and homeomorphisms.

The piecewise linear category, PL, consists of the following: objects are topological manifolds together with a homeomorphism to a simplicial complex. Morphisms are homeomoprhisms which are linear on each simplex.

If we replace "piecewise linear" with "piecewise smooth" we obtain the category PD of piecewise differential manifolds. There is an obvious "forgetful functor" $PL \to PD$ which rise to an equivalence of categories.

For Cat $= TOP$ or PL, the structure set $\mathscr{S}^{\mathrm{Cat}}(X)$ is defined as before except we use Cat manifolds: a structure is a homotopy equivalence $M \xrightarrow{f} X$ where M is a Cat-manifold. The equivalence relation is also adapted to the category in the obvious way. Note that the s- and h-cobordisms theoremms hold for PL (relatively easy) and TOP (much harder).

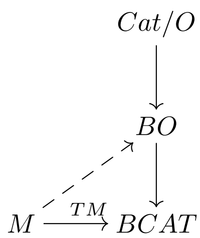
Example 1.5. Consider $\mathscr{S}^{PL}(S^n)$. We next show how the Generalised Poincaré Conjecture (GPC) and smoothing theory imply that for any PL homotopy n-sphere Σ^n, $\Sigma^n_{PL} \cong_{PL} S^n$, $n \geq 5$. It follows that

$$\mathscr{S}^{PL}(S^n) = \{[\mathrm{id}]\}, \quad n \geq 5.$$

If Σ^n_{PL} is smoothable apply the GPC. Now for smoothing theory. There exists spaces BPL and $BTOP$ classifying stable bundles in these categories. Moreover a Cat manifold M has a stable Cat tangent bundle classified by a map $TM \colon M \to B\,\mathrm{Cat}$.

Question: Does M admit a smooth structure?

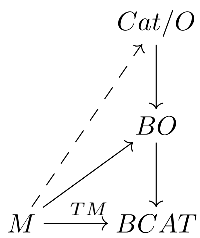
Answer: M is smoothable $\Leftrightarrow$ TM lifts to BO $(n \geq 5)$



A further consequence of smoothing theory. If a Cat-manifold M admits a smooth structure then we can consider the number of vertical homotopy classes of lifts of the classifying map of the Cat tangent bundle. These are classified by maps $[M, \mathrm{Cat}\,/O]$. Focussing on the PL case, we have the following fundamental result from [H-M]: Concordance classes of smooth structures on M^n_{PL} are clasified by the set of homotopy classes $[M, PL/O]$ where

$$M_\alpha \sim_{\text{concordant}} M_\beta$$

if and only if there is a smooth stucture $(M \times I)_\gamma$ such that $(M \times \{0\})_\gamma = M_\alpha$ and $(M \times \{1\})_\gamma = M_\beta$.



Slogan: There is "no L-theory" in smoothing theory:

To obtain the set of diffeomorphism classes of smooth structures on M, one must find the quotient of a certain action of $\pi_0\mathrm{Homeo}(M)$ on $[M, PL/O]$. When $M = S^n$ this action is known to be trivial. Hence:

For S^n concordance classes of smooth structures coincide with diffeomorphism classes.

Theorem 1.6. *For $n \geq 5$, $\Theta_n \cong \pi_n(PL/O)$. For $n \leq 6$, $\pi_n(PL/O) = 0$.*

$$\begin{array}{ccc} & & PL/O \\ & \nearrow & \downarrow \\ & & BO \\ & \nearrow & \downarrow \\ S^n & \xrightarrow{TM} & BPL \end{array}$$

PL/O is an ∞-loop space and hence defines a genearlised cohomology theory.

In particular the set of maps $[M, PL/O]$ is finite since $\pi_i(PL/O)$ is finite for all i. Thus if $n \geq 5$ any PL n-manifold admits finitely diffeomorphism classes of smooth structures.

The homotopy types of G/PL and G/TOP. Let us now return to surgery and look at the consequences of the above for the homotopy groups of G/PL. We have the PL surgery long exact sequence for S^n:

$$\ldots \to \mathscr{S}^{PL}(S^{n+1}) \to [S^{n+1}, G/PL] \to L_{n+1}(\mathbb{Z}) \to \mathscr{S}^{PL}(S^{n)}) \to \ldots$$

Since $\mathscr{S}^{PL}(S^n) = \{[\mathrm{id}]\}$ for $n \geq 5$ we have

$$\pi_i(G/PL) \cong L_i(\mathbb{Z}), \quad i \geq 5.$$

In fact, this holds for $i \geq 1$ with the caveat that that for $i = 4$ the map is multiplication by 2:

$$\mathbb{Z} \cong \pi_4(G/PL) \xrightarrow{\times 2} L_4(\mathbb{Z}) \cong \mathbb{Z}$$

Theorem 1.7 (Kirby-Siebenmann)**.** *There is a homotopy equivalence*

$$TOP/PL = K(\mathbb{Z}/2, 3).$$

Theorem 1.8 (Sullivan for G/PL and Kirby-Siebenmann for G/TOP)**.** *Let $X_{(2)}$ denote localisation of the space X at the prime 2. There are homotopy equivalences*

$$(G/PL)_{(2)} \simeq E \times \prod_{k \geq 2} K(\mathbb{Z}_{(2)}, 4k) \times K(\mathbb{Z}/2, 4k-2)$$

where E is the non-trivial 2-stage Postnikov system

$$K(\mathbb{Z}_{(2)}, 4) \to E \to K(\mathbb{Z}/2, 2)$$

with k-invariant or order two in $H^5(K(\mathbb{Z}/2, 2); \mathbb{Z}) \cong \mathbb{Z}/4$. In addition, there is a homotopy equivalence

$$(G/TOP)_{(2)} \times \prod_{k \geq 1} K(\mathbb{Z}_{(2)}, 4k) \times K(\mathbb{Z}/2, 4k-2).$$

Idea of proof. Let M be closed oriented smoothing manifold with $\Omega_n^{\mathrm{SO}}(G/PL) \ni [M \xrightarrow{\varphi} G/PL]$. Taking id_M as the base-point for $\mathscr{N}(M)$, φ gives rise to the diagram

$$\begin{array}{ccccc} \nu_N & \xrightarrow{\overline{f}} & E & & \\ \downarrow & & \downarrow & & \\ N & \xrightarrow{f} & M & \xrightarrow{\varphi} & G/PL \end{array}$$

where $[\overline{f}, f]$ is the bordism class of degree one normal map prescribed by φ. By [L, Theorem 4.47] the surgery obstruction of $(\overline{f}, f)$ is depends only on the bordism class $[M, \varphi]$ and thus there are well-defined homomorphissm

$$\Omega_n^{\mathrm{SO}}(G/PL) \xrightarrow{\sigma_n} L_n(\mathbb{Z}), \quad [M, \varphi] \mapsto \sigma_{\mathbb{Z}}(\overline{f}, f).$$

Now 2-locally MSO splits as a product of Eilenberg-MacLane spectra and hence

$$\begin{aligned}\Omega_n^{\mathrm{SO}}(G/PL_{(2)}) &\cong \pi_n(\mathrm{MSO}_{(2)} \wedge G/PL_{(2)})\\ &\cong \bigoplus H_*(G/PL_{(2)};\mathbb{Z}) \times H_K(G/PL_{(2)};\mathbb{Z}/2).\end{aligned}$$

In the light of this isomorphism the homomorphisms σ_n give rise to co-homology classes

$$\kappa_i \in H(G/PL;\mathbb{Z}/2) \quad \text{and} \quad l \in H(G/PL:\mathbb{Z}_{(2)}).$$

For $\varphi\colon M \to G/PL$ as above, these cohomology classes satisfy

$$\langle \varphi^* k_n, [M]\rangle = \sigma_{\mathbb{Z}}(N \xrightarrow{f_\varphi} M).$$

Taking $M = S^n$, we see that the product of the corresponding maps $l_{4k} : G/PL \to K(\mathbb{Z}, 4k)$, $\kappa_{4k+2} : G/PL \to K(Z/2, 4k+2)$ gives a map

$$\Pi(l_{4k} \times \kappa_{4k+2})\colon G/PL_{(2)} \to \Pi K(\mathbb{Z}_{(2)}, 4k) \times K(\mathbb{Z}/2, 4k+2)$$

which is an isomorphism on all homotopy groups (except in dimension 4 where it is multiplication by 2). This can be use to identify the 2-local homotopy type of G/PL.

□

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