1. Manifold Atlas : Regensburg Surgery Blockseminar 2012
Spivak normal fibrations (Oliver Straser and Nena Röttgen)

Question. Let X be a topological space. When is X homotopy equivalent to a closed manifold?

Conditions: last talk: Poincaré complex, now: existence of a Spivak normal fibration

Definition 1.1 (Spherical fibration)**.** $p\colon E \to X$ $(k-1)$ spherical fibration $\Leftrightarrow$ p is a fibration such that $F = p^{-1}(x) \simeq S^{k-1}$ for

Example 1.2. $i\colon X \to \mathbb{R}^{n+k}, X$ closed manifold, $\nu(i)\colon \nu(X) \to X$ normal bundle

$$p := S\nu(i) : S\nu(X) := \{v \in \nu(X) \mid ||v|| = 1\} \quad \text{sphere bundle}$$
$$Dp := D\nu(i)\colon D\nu(X) := \{v \in \nu(X) \mid ||v|| \leq 1\} \quad \text{disk bundle}$$

Notation. Let $p : E \to X$ be a spherical fibration. Then

$$D_p\colon DE \to X, DE = \mathrm{cyl}(p)$$

is called the associated disk bundle. and

$$\mathrm{Th}(p) := (DE/E, \infty)$$

its *Thom space.*

Orientation for p. $p\colon E \to X$ $(k-1)$-spherical fibration, X connected, $\gamma\colon I \to X, \gamma(0) = x_0, \gamma(1) = x_1$

1.0.1. *Fiber transport.*

$$\begin{array}{ccc} p^{-1}(x_0) \times \{0\} & \longrightarrow & E \\ \downarrow & \overset{H}{\nearrow} & \downarrow p \\ p^{-1}(x_0) \times I & \xrightarrow[\gamma\circ \mathrm{pr}_2]{} & X \end{array}$$

$\rightsquigarrow f_\gamma := H(\cdot, 1)\colon p^{-1}(x_0) \to p^{-1}(x_1)$

Exercise 1.3.
(1) The homotopy class of f_γ depends only on the homotopy class of γ
(2) $[\gamma] \mapsto [f_\gamma]$ is a homomorphism of monoids

Define: $t_x\colon \pi_1(X, x) \to [p^{-1}(x), p^{-1}(x)]$

Definition 1.4.
- $w\colon \pi_1(X) \to \{\pm 1\}, w(\gamma) = \deg(t(\gamma))$, orientation homomorphism
- p is orientable $\Leftrightarrow$ w is trival.
- p orientable: An orientation of p is a choice $[p^{-1}(x)] \in H_{k-1}(p^{-1}(x))$ f.a. $x \in X$ such that for every path $\gamma : I \to X$ with $\gamma(i) = x_i, i = 0, 1$ holds $H_{k-1}(f_\gamma)([p^{-1}(x_0)]) = [p^{-1}(x_1)]$

Theorem 1.5 (Thom isomorphism)**.** *X connected, finite CW-complex, $p\colon E \to X$ $(k-1)$-spherical fibration. Then there exists a group homomorphism $w\colon \pi_1(X) \to \{\pm 1\}$ and $U_p \in H^k(DE; E; \mathbb{Z}^w)$ such that*

$$H_i(DE, E; \mathbb{Z}^{w/triv}) \xrightarrow{Dp_*(U_p \cap \cdot)} H_{i-k}(X; \mathbb{Z}^{triv/w})$$
$$H^i(X; \mathbb{Z}^{w/triv}) \xrightarrow{Dp^*(\cdot) \cup U_p} H^{i+k}(DE; E;^{triv/w})$$

are isomorphisms. U_p is called the Thom class of p.

Example 1.6. $i\colon X \to \mathbb{R}^{n+k}, X$ closed manifold, $\nu(i)\colon \nu(X) \to X$ normal bundle

$$\begin{aligned}
&p := S\nu(i) : S\nu(X) := \{v \in \nu(X) \mid ||v|| = 1\} \quad \text{sphere bundle}\\
&Dp := D\nu(i)\colon D\nu(X) := \{v \in \nu(X) \mid ||v|| \leq 1\} \quad \text{disk bundle}\\
&c\colon S_{n+k} = \mathbb{R}_{n+k} \cup \{\infty\} \to \mathrm{Th}(p) = D\nu(i)/S\nu(i)\\
&\rightsquigarrow h(c) \in H_{n+k}(D\nu(i), S\nu(i))
\end{aligned}$$

where h is the Hurewicz homomorphism.
$\Rightarrow [X] = Dp_*(U_p \cap h(c))$.

Definition 1.7. Let X be a finite connected n-dimensional Poincaré complex with fundamental class $[X]$. A *Spivak normal fibration* for X is a $(k-1)$-spherical fibration, $p\colon E \to X$ and together with a collapse map $c\colon S^{n+k} \to \mathrm{Th}(p)$ such that

(1) the orientation homomorphisms of p and X coincide
(2) $[X] = Dp_*(U_p \cap h(c))$

Definition 1.8. Let $E_i \to X_i$ be fibrations, a fiber map $(\overline{f}, f)$ is a commutative diagram

$$\begin{array}{ccc}
E_1 & \xrightarrow{\overline{f}} & E_2\\
\downarrow & & \downarrow\\
X_1 & \xrightarrow{f} & X_2
\end{array}$$

Similar we define a fiber homotopy. A fiber homotopy $(\overline{f}_t, f_t)$ is called strong if f_t is constant with respect to t.
If $E_i \to X$ are spherical fibrations, then the *join* of E_1 and E_2 is defined by

$$E_1 * E_2 = I \times (E_1 \times_X E_2)/(e_1, e_2, 0) \sim (e_1', e_2, 0), (e_1, e_2, 1) \sim (e_1, e_2', 1)$$

Let $\underline{S}_k := S^k \times X \to X$, $\sum^k(E) = (\underline{S}^{k-1} * E)$

Theorem 1.9. *Let X be a finite, connected, n-dimensional Poincaré complex.*

(1) *If $k > n$ then there exists a $(k-1)$ Spivak normal fibration*
(2) *$p_i\colon E_i \to X$ are (k_{i-1}) Spivak normal fibration over X with collapse map $c_i\colon S^{n+k_i} \to \mathrm{Th}(p_i)$ for $k \geq k_1, k_2$ exist a fiber map unique up to strong fiber homotopy equivalence*

$$\overline{f}\colon \sum^{k-k_1} p_1 \to \sum^{k-k_2} p_2$$

such that $\pi_{1(\overline{f})=}([\sum^{k-k_1}[c_1]]) = [\sum^{k-k_2} c_2]$

Idea of the proof of (1) in the case $w = 0$.
Fact: There exists an embedding $X \to N \subseteq \mathbb{R}^{n+k}$ $(k > n)$, N regular NHD, s.t.

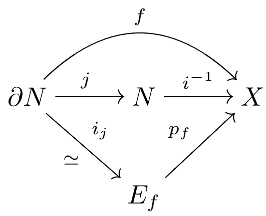
(1) $(N, \partial N)$ is compact, oriented $n+k$ dimensional manifold with boundary.
(2) Let $j : \partial N \to N$ be the canonical inclusion, then $\pi_1(j)$ is an isomorphism and $\pi_2(N, \partial N) = 0$
(3) X is a strong deformation retract of $N \Rightarrow i\colon X \to N$ is a homotopy equivalence.

The proof will be devided into two steps.
Step 1: construct a Thom class
Step 2: $\partial N \to X$ is made into a fibration

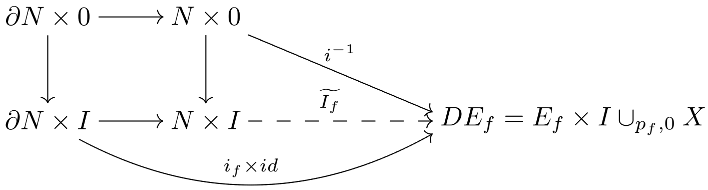
Choose a homotopy inverse i^{-1} of i and $[N, \partial N] \in H_{n+k}(N, \partial N)$ a generator, then for any $u \in H^k(N, \partial N)$ we have a commutative diagram

$$\begin{array}{ccc} H^p(X) & \xrightarrow{\cdot\cap(i^{-1})^*(u\cap[N\partial N])} & H_{n-p}(X) \\ {\scriptstyle u\cup(i^{-1})^*(\cdot)}\downarrow & & \downarrow{\scriptstyle i_*(\cong)} \\ H^{k+p}(N, \partial N) & \xrightarrow{\cap[N,\partial N](\cong)} & H_{n-p}(N) \end{array}$$

There exists exactly one $u \in H^k(N\partial N)$ such that $(i^{-1})^*(u\cap[N, \partial N]) = [X]$. Hence by the diagram above $a \mapsto u \cup (i^{-1})^* a$ is an isomorphism. We fix this u. Every map can be decomposed as follows



suchb that p_f is a fibration and i_f is a homotopy equivalence. Next we want to extend i_f to a homotopy equivalence of pairs $I_f : (N(\partial N)) \to (DE_f, E_f)$ as follows. j is a cofibration $\Rightarrow$ by the homotopy extension property



there exists $\widetilde{I_f}$. Set $I_f := \widetilde{I_f(\ ,1)}$. By definition I_f is homotoic to i^{-1} which is a homotopy equivalence (since $DE_f \simeq X$). So I_f is the desired extension of i_f. We take now the u fixed before to construct our Thom class u_f as follows:

$$u_f := (I_f^*)^{-1}(u) \in H^k(DE_j, E_j)$$

then

$$H(X) \to H(DE_j) \overset{\cup u_f}{\to} H(DE_j, E_j)$$

is an isomorphism. This looks like the Thom isomorphism. In fact one case use the Leray Serre Spectral Sequence to show, that the fiber of the fibration $p_f : E_f \to X$ has to be a homology $k-1$ sphere. Moreover it is not hard to check that the fiber has to be simply connected and so by a theorem of Whitehead p_f is spherical.
Finally let $c : S^{n+k} \to N/\partial N$ be the collapse map introduced before. In the exercises we will show that

$$h(c) = \pm[N, \partial N]$$

If we take $c' := I_f \circ c$ then it is straight foreward to check that

$$(Dp_f)(u_f \cap h(c')) = [X]$$

this was to show.

□

Lemma 1.10. *$p_i \colon E_i \to X_i$ spherical fibrations. X_i Poincaré complex, $(\overline{f}, f) \colon p_1 \to p_2$ which is a homotopy equivalence on each fiber*

(1) $w_2 \circ \mathrm{Th}(f) = w_1$ *(Exercise)*

(2) *Let* $c_1\colon S^{n+k} \to \mathrm{Th}(p_2)$ *be a pointed map.* $c_2 = \mathrm{Th}(\overline{f})$ *then* $\deg f = \pm 1 \Rightarrow (p_1, c_1) SNF \Leftrightarrow (p_2(c_2)) SNF$

(3) *path* p_i *are Spivak normal fibrations* $\Rightarrow \deg f = \pm 1$

Here by $\deg f = \pm 1$ we mean just mean $f_*[X_1] = \pm[X_2]$. Note that f_* is defined in the twisted case only up to a sign.