THE BAND CONNECTED SUM AND THE SECOND KIRBY MOVE FOR HIGHER-DIMENSIONAL LINKS

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Abstract. Let $f : S^q \sqcup S^q \to S^m$ be a link (i.e. an embedding).

How does (the isotopy class of) the knot $S^q \to S^m$ obtained by embedded connected sum of the components of f depend on f?

Define a link $\sigma f : S^q \sqcup S^q \to S^m$ as follows. The first component of σf is the 'standardly shifted' first component of f. The second component of σf is the embedded connected sum of the components of f. How does (the isotopy class of) σf depend on f?

How does (the isotopy class of) the link $S^q \sqcup S^q \to S^m$ obtained by embedded connected sum of the last two components of a link $g : S^q_1 \sqcup S^q_2 \sqcup S^q_3 \to S^m$ depend on g?

We give the answers for the 'first non-trivial case' $q = 4k - 1$ and $m = 6k$. The first answer was used by S. Avvakumov for classification of linked 3-manifolds in S^6.

Contents

1. Introduction

Denote $T^{0,q} := S^0 \times S^q = S^q \sqcup S^q$. Take a link, i.e. an embedding $f : T^{0,q} \to S^m$. (Up to isotopy this is equivalent to taking two numbered oriented q-spheres in S^m.)

Make embedded connected sum of the components of f along some tube (=band) joining them, i.e. along some embedding $h : [-1, 1] \times S^{q-1} \to S^m$ such that

$$h(\pm 1 \times S^{q-1}) \subset f(\pm 1 \times S^q) \quad \text{and} \quad f(T^{0,q}) \cap h((-1, 1) \times S^{q-1}) = \emptyset.$$

For $m \geq q + 3$ the isotopy class $\#[f]$ of this connected sum is independent of the choices of the tube, and of the link f within its isotopy class $[f]$.[1]

How does the class $\#[f]$ depend on $[f]$?

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[1] The proof is obtained either using [Sk15, the Standardization Lemma 2.1] or analogously to [Sk11, Lemma 3.2], cf. [Ha66A, Theorem 1.7], [Av17, §1]. Analogous assertion is false for $m = q + 2$, e.g. for $m = q + 2 = 3$. For $m = q + 2 = 3$ this operation is called *band-connected sum* of the components of the link. Unlike in this paper, this operation was mostly studied for *split* links, for which the components are contained in disjoint cubes.

• For $2m \ge 3q+4$ every two embeddings $S^q \to S^m$ are isotopic, by a result of Haefliger-Zeeman [Sk16c, Unknotting Spheres Theorem 2.3], [Sk06, Theorem 2.7.a,b].

• We give the answer for the 'first non-trivial case' $2m = 3q + 3$ (under additional assumption that m is even, see Connected Sum Theorem 2.2).

This answer was used by S. Avvakumov for classification of linked 3-manifolds in S^6 [Av14, Av17]. The answer gives an alternative construction of a generator in the group of knots $S^{4k-1} \to S^{6k}$ for $k = 1, 3, 7$ (Corollary 2.3.b). Another applications are the following.

Take a 3-component link, i.e. an embedding $g : S^q_1 \sqcup S^q_2 \sqcup S^q_3 \to S^m$. Make embedded connected sum of the second and the third components of g along some tube joining them. We obtain a link $\#_{23}g : T^{0,q} \to S^m$. Analogously to footnote 1 for $m \ge q + 3$ the isotopy class $\#_{23}[g]$ of $\#_{23}g$ is independent of the choices of the tube, and of the link g within its isotopy class $[g]$.

How does $\#_{23}[g]$ depend on $[g]$?

• For $2m \ge 3n + 4$ the linking coefficient λ_{21} of $\#_{23}g$ (which is defined in §2 and which completely defines the isotopy class of $\#_{23}g$ by another result of Haefliger-Zeeman [Sk16h, Theorem 4.1], [Sk06, Theorem 3.1]) equals the sum $\lambda_{21}(g) + \lambda_{31}(g)$ of the linking coefficients of g.

• We describe $\#_{23}[g]$ for the 'first non-trivial case' $2m = 3q + 3$ (under additional assumption that m is even, see Theorem 2.5 and Proposition 2.1.c).

Define a higher-dimensional *unframed second Kirby move* σ as follows, cf. [Ma80, §3.1]. The first component of the link $\sigma f : T^{0,q} \to S^m$ is the 'standardly shifted' (see the details in §2) first component of a link $f : T^{0,q} \to S^m$. The second component of the link σf is the embedded connected sum of the components of f along some tube joining them. For $m \ge q + 3$ the isotopy class $\sigma[f]$ of σf is independent of the choices of the tube, and of the link f within its isotopy class $[f]$ [Sk11, Lemma 3.2].

How does $\sigma[f]$ depend on $[f]$?

• For $2m \ge 3q+4$ we have $\sigma[f] = [f]$ (because the isotopy class of a link is completely defined by λ_{21}, which is preserved by σ).

• We give the answer for the 'first non-trivial case' $2m = 3q + 3$ (under additional assumption that m is even, see Theorems 2.6 and 2.2, Proposition 2.1.c).

As byproducts we

• show that $\sigma \ne \pm\,\mathrm{id}$ (Corollary 2.7 and the text below), and

• obtain an alternative classification of links $T^{0,4k-1} \to S^{6k}$, thus proving one version of the conjecture for $p = 0$ in [Sk15, Remark 1.9.b], and disproving another version (Corollary 2.8).

This is interesting because this rules out a natural inductive proof of classification of embeddings $S^p \times S^q \to S^m$ [Sk15, Conjecture 1.3 and Remark 1.9.c].

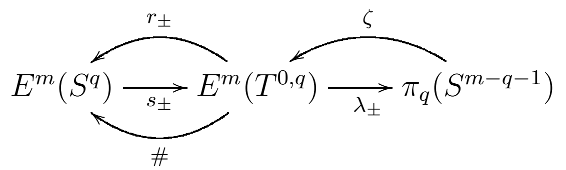
2. Statements of main results

We work in the smooth category which we omit from the notation. Analogues of Theorem 2.5.abc, 2.6, and Remark 5.2 for the PL category are correct, cf. [Sk16h, §6]. For a manifold N denote by $E^m(N)$ the set of embeddings $N \to \mathbb{R}^m$ up to isotopy. By $[\cdot]$ we denote the isotopy class of an embedding or the homotopy class of a map. We assume that $m \ge q + 3$, unless indicated otherwise.

The sum operations on $E^m(T^{0,q})$ and on $E^m(S^q)$ are 'embedded connected sums of two embeddings whose images are contained in disjoint cubes'. See accurate definition of abelian group structures on these sets in [Ha66A, Ha66C].

Identify $E^{6k}(S^{4k-1})$ with $\mathbb{Z}$ by the isomorphism of [Ha62k, Ha66A, Sk06'].

Below we define and use the following diagram of groups and homomorphisms.



The map # defined in §1 is clearly a homomorphism.

Definition of $r_\pm$ and $\lambda_\pm$. Let $r_\pm$ be 'the knotting class of the component', i.e. $r_\pm$ is induced by the inclusion $\pm 1 \times S^q \subset T^{0,q}$, where the orientation on $\pm 1 \times S^q$ corresponds to the standard orientation on S^q.

Let $\lambda_\pm$ be the linking coefficient, i.e. the homotopy class of $f|_{\pm 1 \times S^q}$ in the complement to the other component, see accurate definition in [Sk16h, §3], [Sk06, §3].

Let

$$H : \pi_{4k-1}(S^{2k}) \to \mathbb{Z}$$

be the Hopf invariant (defined to be the linking number of preimages of two regular points under a smooth or a PL approximation of a map $S^{4k-1} \to S^{2k}$).

The following result is essentially due to Haefliger, see proof in §5.

Proposition 2.1. *(a) The following map has a finite kernel:*

$$H\lambda_+ \oplus H\lambda_- \oplus r_+ \oplus r_- : E^{6k}(T^{0,4k-1}) \to \mathbb{Z}^4.$$

(b) The image of this map is $\{(a,b,c,d) \ : \ a \equiv b \mod 2\}$ *for* $k = 1, 3, 7$*, and is* $\{(a,b,c,d) \ : \ a \equiv b \equiv 0 \mod 2\}$ *otherwise.*

(c) The following map is a monomorphism

$$H\lambda_+ \oplus \lambda_- \oplus r_+ \oplus r_- : E^{6k}(T^{0,4k-1}) \to \mathbb{Z} \oplus \pi_{4k-1}(S^{2k}) \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

(d) The image of this map is $\{(a,b,c,d) \ : \ a \equiv Hb \mod 2\}$ *for* $k = 1, 3, 7$*, and is* $\{(a,b,c,d) \ : \ a \equiv Hb \equiv 0 \mod 2\}$ *otherwise.*

Theorem 2.2 (Connected Sum; proved in §3)**.** *For $q = 4k-1$ and $m = 6k$ we have*

$$\# = r_+ + r_- \pm \frac{H\lambda_+ + H\lambda_-}{2}.$$

The sign in this formula (and in Corollary 2.3.a, Theorem 2.5) could depend on k.

The integers $H\lambda_+$ and $H\lambda_-$ have the same parity by Proposition 2.1.b.

Our proof of Connected Sum Theorem 2.2 (and of Corollary 2.3, Theorem 2.6.b below) is not by directly using definition of the isomorphism $E^{6k}(S^{4k-1}) \to \mathbb{Z}$ (or of the map $H\lambda_+ : E^{6k}(T^{0,4k-1}) \to \mathbb{Z}$). It would be interesting to obtain a direct proof.

Definition of the standard embedding $\mathrm{i}_{m,q}$. The natural normal framing on the standard embedding $S^q \to S^m$ defines the standard embedding $\mathrm{i}_{m,q} : D^{m-q} \times S^q \to S^m$.

Definition of the Zeeman map ζ. For a map $x : S^q \to S^{m-q-1}$ let $\overline{\zeta}_x : S^q \to S^m$ be the composition

$$S^q \overset{x \times \mathrm{id}\, S^q}{\to} S^{m-q-1} \times S^q \overset{\mathrm{i}_{m,q}}{\to} S^m.$$

We have $\overline{\zeta}_x(S^q)\cap \mathrm{i}_{m,q}(0\times S^q)\subset \mathrm{i}_{m,q}(S^{m-q-1}\times S^q\cap 0\times S^q)=\emptyset$. Define an embedding $g_x:T^{0,q}\to S^m$ by $g_x(1,a):=\overline{\zeta}_x(a)$ and $g_x(-1,a)=\mathrm{i}_{m,q}(0,a)$. Define $\zeta[x]:=[g_x]$.

Clearly, ζ is well-defined and is a homomorphism, see [Ha66C, Theorem 10.1], [Sk11, Definition of Ze before Lemma 3.4].

Corollary 2.3. *(a) We have $\#\zeta=\pm H$ on $x\in\pi_{4k-1}(S^{2k})$.*

(b) For any $k=1,2,4$ let $\eta\in\pi_{4k-1}(S^{2k})$ be the homotopy class of the Hopf map. The embedded connected sum $\#\zeta\eta$ of the components of $\zeta\eta$ is a generator of $E^{6k}(S^{4k-1})\cong\mathbb{Z}$.

Part (b) follows from (a) because $H\eta=1$. Part (a) follows by Connected Sum Theorem 2.2 and Lemma 2.4 below since $r_\pm\zeta=0$ and $\lambda_+\zeta=\mathrm{id}\,\pi_q(S^{m-q-1})$ (the latter is simple and known, see [Ha66C, Proof of Theorem 10.1], [Sk16h, Remark 3.2.c]).

Lemma 2.4 (proved in §3)**.** *We have $H\lambda_-\zeta=H$ on $\pi_{4k-1}(S^{2k})$.*

Definition of r_i,λ_{ij},h_{ij} and μ for an embedding $g:S^q_1\sqcup S^q_2\sqcup S^q_3\to S^m$. Let $r_i=r_i(g)\in E^m(S^q)$, $i\in[3]$, be the isotopy classes of the restrictions of g to the components. Let

$$\lambda_{ij}=\lambda_{ij}(g)\in\pi_q(S^{m-q-1}),\quad (i,j)\in[3]^2,\quad i\ne j,$$

be the pairwise linking coefficients of the components ($\lambda_{ij}=\lambda_+(g|_{S^q_i\sqcup S^q_j})$ is the class of the i-th component in the complement of the j-th component). For $m=6k$ and $q=4k-1$ denote

$$h_{ij}=h_{ij}(g)=\frac{1}{2}(H\lambda_{ij}+H\lambda_{ji}).$$

Let $\lambda^1=\lambda^1(g)\in\pi_q(S^{m-q-1}\vee S^{m-q-1})$ be homotopy class of $g|_{S^q_1}$ in $S^m-g(S^q_2\sqcup S^q_3)$, see accurate definition in [Ha62l, §4], [HS64], [Ha66C, proof of Theorem 9.4]. For $3m\ge 4q+6$ the *triple linking coefficient* $\mu=\mu(g)$ is defined to be the image of λ^1 under the composition

$$\pi_q(S^{m-q-1}_2\vee S^{m-q-1}_3)\to\frac{\pi_q(S^{m-q-1}_2\vee S^{m-q-1}_3)}{i_{2*}\pi_q(S^{m-q-1}_2)\oplus i_{3*}\pi_q(S^{m-q-1}_3)}\to\pi_q(S^{2m-2q-3})\overset{\Sigma^\infty}{\to}\pi^S_{3q-2m+3}$$

of the projection from the Hilton theorem and the stable suspension (since $3m\ge 4q+6$, we have $q<3(m-q)-5$, so the second map is an isomorphism; since $3m\ge 4q+6$, we have $q\le 2(2m-2q-3)$, so the stable suspension Σ^∞ is an isomorphism; μ was denoted by λ^1_{23} in [Ha62l, §4]).

Theorem 2.5. *For $q=4k-1$ and $m=6k$ we have*

(a) $\lambda_-\#_{23}=\lambda_{21}+\lambda_{31}$ and $r_+\#_{23}=r_1$.

(b) $r_-\#_{23}=r_2+r_3\pm h_{23}$.

(c) $H\lambda_+\#_{23}=2\mu+H\lambda_{12}+H\lambda_{13}$.

(d) $\#\#_{23}=r_1+r_2+r_3\pm(\mu+h_{12}+h_{23}+h_{31})$.

Here (a) is obvious (and holds for $m-q\ge 3$), (b) holds by Connected Sum Theorem 2.2, and (c,d) are non-trivial (they are proved in §4 using interpretation of linking coefficients via Pontryagin construction, see Lemmas 3.2 and 4.1).

Definition of the unframed second Kirby move $\sigma:\ker r_+\to\ker r_+$. Represent an element of $\ker r_+$ by an embedding $f:T^{0,q}\to S^m$ such that $f=\mathrm{i}_{m,q}$ on $1\times S^q$ and $f(-1\times S^q)\cap\mathrm{i}_{m,q}(D^1\times S^q)=\emptyset$. Define embedding $g:T^{0,q}\to S^m$ as $g(1,x)=\mathrm{i}_{m,q}(0,x)$, and as the embedded connected sum $f|_{1\times S^q}\#f|_{-1\times S^q}$ of the components of f, with parallel orientations, on $-1 \times S^q$. Define σ by $\sigma[f] := [g]$. The map σ is well-defined and is a homomorphism [Sk11, Lemmas 3.1–3.3].

Comment. The map σ is an isomorphism. This follows from [Sk11, Theorem 1.6] because the maps i^* in the exact sequence are surjections. A simple direct proof is as follows. Define $\sigma' : \ker r_+ \to \ker r_+$ analogously to σ but taking embedded connected sum of $f|_{1\times S^q}$ with reversed orientation and $f|_{-1\times S^q}$. The second component of a representative of $\sigma'\sigma[f]$ is obtained from the second component of f by adding the two first components with different orientations, which cancel outside the shifted first component. Therefore $\sigma'\sigma = \mathrm{id}$. Analogously $\sigma\sigma' = \mathrm{id}$. Thus σ' is the inverse of σ.

Theorem 2.6. *(a) We have $r_-\sigma = \#$ and $\lambda_-\sigma = \lambda_-$.*
(b) For $q = 4k-1$ and $m = 6k$ we have $H\lambda_+\sigma = H\lambda_+ + 2H\lambda_-$.

Here
- the formula for $r_-\sigma$ is obvious;
- the formula for $\lambda_-\sigma$ follows since in the definition of σ the restrictions of g and f to the second component $-1 \times S^q$ are homotopic as maps to $S^m - f(1 \times S^q)$;
- part (b) is non-trivial and is proved in §5.

Comment. Theorems 2.5.c and 2.6.b are particularly interesting because these are essentially PL results (indeed, the linking coefficients and the second Kirby move can be defined in the PL category) proved using differential topology. (It is not clear how to prove Theorems 2.5.c and 2.6.b directly in the PL category because its is not clear how to calculate directly $H\lambda_+\#_{23}$ and $H\lambda_+\sigma$; the calculations in §§3,4,5 use Connected Sum Theorem 2.2, whose PL version is an 'empty' result because any two PL embeddings $S^{4k-1} \to \mathbb{R}^{6k}$ are PL isotopic.)

Definition of $s_\pm$. For an embedding $g : S^q \to S^m$ let $s_\pm(g)$ be any link whose components are contained in disjoint balls, whose restriction to $\pm 1 \times S^q$ is g and whose restriction to the other component is the standard embedding. Define $s_\pm[g] := [s_\pm(g)]$.

Corollary 2.7 (proved in §5). *If $q = 4k-1$ and $m = 6k$, then $\mathrm{im}(\sigma\zeta - \zeta) \not\subset \mathrm{im}(s_+ \oplus s_-)$. In other words, there is $x \in \pi_{4k-1}(S^{2k})$ such that no link representing $\sigma\zeta x - \zeta x$ is piecewise-smoothly isotopic to the standard link.*

The analogue of Corollary 2.7 for $\sigma\zeta + \zeta$ instead of $\sigma\zeta - \zeta$ follows just because $\lambda_-\sigma = \lambda_-$ by Theorem 2.6.a. Corollary 2.7 is non-trivial because of $\lambda_-\sigma = \lambda_-$ (Theorem 2.6.a) and Lemma 5.5.a.

Definition of $\psi_\pm$. Let $\psi_\pm$ be the 'change of the orientation of $\pm 1 \times S^q$' self-map of $E^m(T^{0,q})$.

Comment. The map ψ_+ is described for $q = 4k-1$ and $m = 6k$ by Symmetry Lemma 5.1.b and Proposition 2.1.c. A description of ψ_- is analogous. Observe that $\#\psi_+\zeta = 0$ for any $m \geq q+3$ (as opposed to $\#\zeta \neq 0$, see Corollary 2.3) because a representative of $\#\psi_+\zeta$ spans a ball in $\mathbb{R}^m$. The map σ' of the Comment before Theorem 2.6 equals $\psi_+\sigma\psi_+$.

Corollary 2.8 (proved in §5). *Assume that $q = 4k-1$ and $m = 6k$.*
(a) The following map is an isomorphism:

$$\psi_- r \oplus \sigma|_K \oplus s_- : E^m(D^1 \times S^q) \oplus K \oplus E^m(S^q) \to E^m(T^{0,q}), \qquad \textit{where}$$

• *r is the restriction map, where the inclusion $T^{0,q} \to D^1 \times S^q$ is the boundary inclusion (as opposed to the product inclusion), i.e. the orientations on the components of $T^{0,q}$ are the boundary orientations (as opposed to the product orientations),*

• *$K := \ker(\lambda_+ \oplus r_+ \oplus r_-)$, and*

• *the sum operation on $E^m(D^1 \times S^q)$ is 'embedded connected sum of q-spheres together with normal vector fields', see detailed construction in* [Sk15, §2.1].

(b) The following map is not surjective:

$$r \oplus \sigma|_K \oplus s_- : E^m(D^1 \times S^q) \oplus K \oplus E^m(S^q) \to E^m(T^{0,q}).$$

Concerning low-dimensional version of $\#$ and σ see Remark 5.6.

3. Proof of Theorem 2.2 and Lemma 2.4

Proof of Connected Sum Theorem 2.2. Clearly, $\#, r_+, r_-, \lambda_+, \lambda_-$ are homomorphisms. The group $E^{6k}(T^{0,4k-1})$ is generated by (isotopy classes of) links whose components are contained in disjoint smooth balls, and by $K_0 := \ker(r_+ \oplus r_-)$. We have $\# = r_+ + r_-$ for the former links. Hence it suffices to prove the theorem for links in K_0.

By Proposition 2.1.a the map $H\lambda_+ \oplus H\lambda_- : K_0 \to \mathbb{Z}^2$ has finite kernel. Since any homomorphism from a finite group to $\mathbb{Z}$ is zero, this kernel goes to 0 under the map $\#$. Hence $\#|_{K_0} = n \circ (H\lambda_+ \oplus H\lambda_-)$ for some homomorphism $n : \mathrm{im}(H\lambda_+ \oplus H\lambda_-) \to \mathbb{Z}$. So $\#|_{K_0} = n_+ H\lambda_+ + n_- H\lambda_-$ for some $n_\pm = n_{\pm,k} \in \mathbb{Q}$. Analogously to the commutativity of summation on $E^m(S^q)$ [Ha66A, §1.4], $\#$ is invariant under exchange of the components. Hence $n_+ = n_-$. So by the following Whitehead Link Lemma 3.1 $n_+ = \pm 1/2$. ☐

Lemma 3.1 (Whitehead Link)**.** *For any $l \geq 2$ there is an embedding $\omega : T^{0,2l-1} \to S^{3l}$ such that*

$$r_+\omega = r_-\omega = 0, \quad \lambda_-\omega = 0, \quad \textit{and, for } l \textit{ even}, \quad H\lambda_+\omega = \pm 2, \quad \#\omega = 1.$$

Proof of Lemma 3.1 except $H\lambda_+\omega = \pm 2$. (The following construction of Borromean rings and their spanning disks is known [Ha62k, §4], and the proof modulo this construction is not hard.)

Recall that isotopy classes of embeddings $S^q \to S^n$ are in 1–1 correspondence with h-cobordism classes of oriented submanifolds of S^n diffeomorphic to S^q for $n \geq 5$, cf. [Ha66A, 1.8].

Denote coordinates in $\mathbb{R}^{3l} \subset S^{3l}$ by $(x, y, z) = (x_1, \ldots, x_l, y_1, \ldots, y_l, z_1, \ldots, z_l)$. The Borromean rings is the embedding whose image is disjoint union $S_1 \sqcup S_2 \sqcup S_3 \to \mathbb{R}^{3l}$ of the three $(2l-1)$-spheres given by the following three systems of equations

$$\begin{cases} x = 0 \\ |y|^2 + 2|z|^2 = 1 \end{cases}, \qquad \begin{cases} y = 0 \\ |z|^2 + 2|x|^2 = 1 \end{cases} \qquad \text{and} \qquad \begin{cases} z = 0 \\ |x|^2 + 2|y|^2 = 1 \end{cases}.$$

The embedding (up to isotopy) is defined by taking the orientations on the components as described in [Ha62k, §4].

Let $\omega : T^{0,2l-1} \to \mathbb{R}^{3l}$ be an embedding such that $\omega_{1\times S^{2l-1}}$ is (defined up to isotopy as) oriented S_1, and $\omega|_{-1\times S^{2l-1}}$ is embedded connected sum of oriented S_2 and S_3 along some tube $\tau \cong \partial D^{2l-1} \times I$ joining S_2 and S_3.

For l even $\#\omega = 1$ by [Ha62k, §4].

In this paragraph we prove that *for each $i, j \in \{1, 2, 3\}$, $i \neq j$, there are disjoint $2l$-disks $D_{ij}, D_{ji} \subset \mathbb{R}^{3l}$ bounded by S_i and S_j, respectively.* By symmetry, it suffices to prove this for $i=2$, $j=3$. Take $2l$-disks $D_{23}, D_{32} \subset \mathbb{R}^{3l}$ given by the equations

$$\begin{cases} y=0 \\ |z|^2+2|x|^2 \le 1 \end{cases} \qquad \text{and} \qquad \begin{cases} z_1 \ge 0 \\ z_2 = \ldots = z_l = 0 \\ |x|^2+2|y|^2+\frac{1}{2}|z|^2 = 1 \end{cases}.$$

These disks are bounded by S_2 and S_3, respectively. On the intersection $D_{23} \cap D_{32}$ we have $2 = 2|x|^2 + |z|^2 \le 1$, hence $D_{23} \cap D_{32} = \emptyset$.

Clearly, $r_+\omega = 0$. Since the spheres S_2 and S_3 bound disjoint embedded $2l$-disks D_{23} and D_{32}, we have $r_-\omega = 0$.

Take oriented embedded boundary connected sum of D_{21} and D_{31} by a half-tube $D^{2l-1}\times I$ disjoint from S_1, and such that $\partial D^{2l-1} \times I = \tau$. We obtain a self-intersecting $2l$-disk bounded by $\omega(-1\times S^{2l-1})$ and disjoint from S_1. Then $\lambda_-\omega = 0$. (An informal explanation for $\lambda_-\omega = 0$ is that by making self-intersection of the last two of the Borromean rings, we can drag them apart from the first ring.) □

We use Pontryagin isomorphism between $\pi_q(S^n)$ and the set of framed cobordism classes of framed $(q-n)$-submanifolds of S^q [Pr06, §18.5].

Lemma 3.2. *(a) Let $f : T^{0,q} \to S^m$ be a link whose restrictions to the components are isotopic to standard embedding. The class $\lambda_+ f$ goes under Pontryagin isomorphism to the framed intersection of an arbitrarily framed $f(1 \times S^q)$ and a general position arbitrarily framed $(q+1)$-disk spanned by $f(-1 \times S^q)$.*

(b) Denote by $H\alpha$ the Hopf invariant of a framed $(2k-1)$-submanifold α of S^{4k-1}. If framed $(2k-1)$-submanifolds α, β of S^{4k-1} are disjoint, then $H(\alpha \sqcup \beta) = H\alpha + H\beta + 2\,\mathrm{lk}(\alpha,\beta)$.

Part (a) is proved analogously to the particular case [Av17, Lemma 4.1]. (Although the statement of [Av17, Lemma 4.1] involved Hopf invariant, the proof calculated $\lambda_+ f$ not $H\lambda_+ f$.) Part (b) follows because $H\alpha$ is the linking number of α and the shift of α along the first vectors of the framing. Although part (b) is not published, and part (a) is not published before [Av17], both parts are presumably folklore results known before [Av17].

Proof that $H\lambda_+\omega = \pm 2$ for l even. (This was stated without proof in [Ha62l, end of §6], and the proof presented below is not hard.)

Take oriented embedded boundary connected sum of D_{23} and D_{32} by a half-tube $D^{2l-1}\times I$ disjoint from S_1, and such that $\partial D^{2l-1}\times I = \tau$. We obtain an embedded $2l$-disk bounded by $\omega(-1\times S^{2l-1})$. Its intersection with S_1 is $\Sigma_1 := (D_{23}\cap S_1)\sqcup(D_{32}\cap S_1)$. The intersection $D_{23} \cap S_1$ is transversal, and is the $(l-1)$-sphere given by $x = y = 0$, $|z|^2 = \frac{1}{2}$. The intersection $D_{32} \cap S_1$ is transversal, and is the $(l-1)$-sphere given by

$$x = z_2 = \ldots = z_l = 0, \quad z_1 = \sqrt{\frac{2}{7}}, \quad |y|^2 = \frac{3}{7}.$$

We have

$$H\lambda_-\omega \overset{(1)}{=} H\Sigma_1 \overset{(2)}{=} 2\,\mathrm{lk}_{S_1}(D_{23}\cap S_1, D_{32}\cap S_1) \overset{(3)}{=} \pm 2, \quad \text{where}$$

- Σ_1 is considered as a framed intersection, for some framings on D_{23}, D_{32} and S_1;
- equality (1) holds by Lemma 3.2.a;
- $D_{23} \cap S_1$ and $D_{32} \cap S_1$ are considered as oriented intersections;
- equality (2) holds by Lemma 3.2.b because S_1 and S_2 bound disjoint disks, S_1 and S_3 bound disjoint disks, so by Lemma 3.2.a both framed intersections $D_{23}\cap S_1$ and $D_{32}\cap S_1$ are framed cobordant to zero.

Let us prove equality (3). The $(l-1)$-sphere $D_{23}\cap S_1$ bounds in S_1 the l-disk Δ given by

$$x=y_2=\ldots=y_l=0,\quad y_1\ge 0,\quad y_1^2+2|z|^2=1.$$

The intersection of Δ and $D_{32}\cap S_1$ is the only point

$$x=y_2=\ldots=y_l=z_2=\ldots=z_l=0,\quad y_1=\sqrt{\frac{3}{7}},\quad z_1=\sqrt{\frac{2}{7}}.$$

This is a transversal intersection point. This implies equality (3). $\square$

Sketch of an alternative proof of Connected Sum Theorem 2.2 for $k=1$. Take a representative $f:T^{0,3}\to S^6$ of an isotopy class from $E^6(T^{0,3})$. Denote by g a representative of $\#[f]$. The formula follows by [Wa66, Theorem 4] because 'the homology class of handle' $g_0\in H_4(M_g)$ 'goes to' f_++f_-, so $[g]=(f_++f_-)^3/6$. $\square$

Let $\iota_n\in\pi_n(S^n)$ be the homotopy class of the identity map.

Lemma 3.3 (well-known)**.** *For any $x\in\pi_{4k-1}(S^{2k})$ we have $H((-\iota_{2k})\circ x)=Hx$.*

Proof. Change of the orientation of S^{2k} changes the orientations on preimages of regular points under a map $S^{4k-1}\to S^{2k}$. Change of the orientation of both components preserves the linking number. Hence change of the orientation of S^{2k} preserves the Hopf invariant. $\square$

Proof of Lemma 2.4. For a map $y:S^q\to S^{m-q-1}$ the link obtained from $\overline{\zeta}_y$ by exchange of components is isotopic to $\overline{\zeta}_{S\circ y}$, where S is the symmetry of S^{m-q-1} w.r.t the origin. Then $\lambda_-\zeta[y]=((-1)^{m-q}\iota_{m-q-1})\circ[y]$. So $H\lambda_-\zeta[y]=H((-\iota_{2k})\circ[y])=H[y]$ by Lemma 3.3. $\square$

4. Proof of Theorems 2.5.c,d

Lemma 4.1. *Let*
- *$g:S_1^{2l-1}\sqcup S_2^{2l-1}\sqcup S_3^{2l-1}\to S^{3l}$ be an embedding such that $\lambda_{23}(g)=\lambda_{32}(g)=0$, and*
- *$D_2,D_3\subset S^{3l}$ be disjoint oriented embedded $2l$-disks in general position to $g_1:=g|_{S_1^{2l-1}}$,*

and such that $g(S_j^{2l-1})=\partial D_j$ for each $j=2,3$.

Then for $j=2,3$ the oriented preimage $g_1^{-1}D_j$ is a closed oriented $(l-1)$-submanifold of S_1^{2l-1} missing $g_1^{-1}D_{5-j}$, and $\mu(g)$ equals the linking number of $g_1^{-1}D_2$ and $g_1^{-1}D_3$ in S_1^{2l-1}.

This holds by the well-known 'linking number' definition of the Hopf-Whitehead invariant $\pi_{2l-1}(S^l\vee S^l)\to\pi_3(S^3)\cong\mathbb{Z}$ [Sk20e, §2, Sketch of a proof of (b1)].

Proof of Theorem 2.5.c,d. Clearly, $\#_{23},r_i,\lambda_{ij}$ are homomorphisms. Also μ is a homomorphism. We have $\lambda_+\#_{23}=0$ and $\#\#_{23}=r_1+r_2+r_3$ for links whose components are contained in pairwise disjoint smooth balls. Hence (analogously to the proof of Theorem 2.2) it suffices to prove (c,d) for links in $K_0:=\ker(r_1\oplus r_2\oplus r_3)$.

If $\lambda_{23}g=\lambda_{32}g=0$, then by Proposition 2.1.c the spheres $g(S_2^{4k-1})$ and $g(S_3^{4k-1})$ bound disjoint embedded $4k$-disks. Then (c) follows by Lemmas 3.2.ab and 4.1.

The sum

$$\mu\oplus\sum_{(i,j)\in[3]^2,\ i\ne j}\lambda_{ij}:K_0\to\mathbb{Z}\oplus\pi_{4k-1}(S^{2k})^6$$

This sum has a finite kernel by [Ha66C, Theorem 9.4] and [CFS, Lemma 1.3], see also [Sk16h, Theorem 9.3.b]. (Note that this sum is a monomorphism for $k>1$ by [Ha62l, Theorem in §6], [Ha66C, Theorem 9.4], see also [Sk16h, Remarks 8.2ab and Theorem 8.3].) Since any homomorphism from a finite group to $\mathbb{Z}$ is zero, (analogously to the proof of Theorem 2.2) by Proposition 2.1.a and the case $\lambda_{23} = \lambda_{32} = 0$ of (c) we have

$$H\lambda_+\#_{23}|_{K_0} = 2\mu + l_{23}H\lambda_{23} + l_{32}H\lambda_{32} + H\lambda_{12} + H\lambda_{13} \quad \text{for some} \quad l_{23}, l_{32} \in \mathbb{Q}.$$

Analogously to the commutativity of summation on $E^m(S^q)$ [Ha66A, §1.4], $\#_{23}$ is invariant under exchange of the second and the third components. By [HS64, Theorem in p. 259] μ is invariant under any permutation of all the three components.[2] Hence $l_{23} = l_{32}$.

Then by Connected Sum Theorem 2.2

$$\#\#_{23}|_{K_0} = \pm h_{23} \pm \frac{1}{2}(H\lambda_{21} + H\lambda_{31} + 2\mu + 2l_{23}h_{23} + H\lambda_{12} + H\lambda_{13}).$$

Under any permutation of all the three components both $\#\#_{23}$ and μ remain the same. Hence $l_{23} = 0$. □

5. Proof of Theorem 2.6.b and Corollaries 2.7, 2.8

Part (a) of the following lemma should be compared to [Sk05, §3, Symmetry Remark] where situation is 'the opposite'.

Lemma 5.1 (Symmetry). *(a) For any embedding $g : S^{4k-1} \to S^{6k}$ the composition with the reflection-symmetry of S^{6k} is isotopic to g.*

Or, equivalently, for any embedding $g : S^{4k-1} \to S^{6k}$ the composition with the reflection-symmetry of S^{4k-1} represents a knot $-[g] \in E^{6k}(S^{4k-1})$.

(b) We have $r_-\psi_+ = r_-$, $r_+\psi_+ = -r_+$, $\lambda_+\psi_+ = -\lambda_+$, and $H\lambda_-\psi_+ = H\lambda_-$.

Proof. (a) This follows by definition of the Haefliger isomorphism $E^{6k}(S^{4k-1}) \to \mathbb{Z}$ [Ha62l, §2], [Sk16s, §3].

(b) The equation $r_-\psi_+ = r_-$ is clear. The equation $r_+\psi_+ = -r_+$ holds by (a). We have $\lambda_+\psi_+ = \lambda_+ \circ (-\iota_{4k-1}) = -\lambda_+$. We have $H\lambda_-\psi_+ = H((-\iota_{2k}) \circ \lambda_-) = H\lambda_-$ by Lemma 3.3. □

Proof of Theorem 2.6.b. Theorem 2.6.b follows because on $\ker r_+$ we have

$$2r_- \overset{(1)}{=} 2\#\psi_+\sigma \overset{(2)}{=} (2r_- \pm H\lambda_+ \pm H\lambda_-)\psi_+\sigma \overset{(3)}{=} (2r_- \mp H\lambda_+ \pm H\lambda_-)\sigma \overset{(4)}{=}$$
$$= 2\# \mp H\lambda_+\sigma \pm H\lambda_- \overset{(5)}{=} 2r_- \pm H\lambda_+ \pm H\lambda_- \mp H\lambda_+\sigma \pm H\lambda_-, \quad \text{where}$$

- equality (1) holds because two copies of the first component having opposite orientations 'cancel';
- equalities (2) and (5) hold by Connected Sum Theorem 2.2 because $r_+ = 0$, $r_+\sigma = 0$, so $r_+\psi_+\sigma = 0$ (because change of the orientation of the *standard* embedding $S^q \to \mathbb{R}^m$ gives embedding $S^q \to \mathbb{R}^m$ isotopic to the standard one);
- equality (3) holds by Symmetry Lemma 5.1.b;
- equality (4) holds by Theorem 2.6.a. □

Sketch of an alternative proof of Theorem 2.6.b for $k = 1$. Take a representative $f : T^{0,4k-1} \to S^{6k}$ of an element from $\ker r_+$. Take a representative g of $\sigma[f]$. Analogously to [Wa66, §4] there is a unique framing of f such that $p_k(M_f) = 0$ for the $6k$-manifold M_f obtained from S^{6k} by surgery along f with this framing. Denote by $f_\pm \in H_{4k}(M_f)$ 'the homology classes

[2] Clearly, there is a typo in [Ha62l, §6, Theorem, (2)] because the sign could not depend on the numbering. Clearly, there is a typo in [HS64, Theorem in p. 259]: p_1, p_2, p_3 should be i, j, k, respectively. Hence [Ha62l, §6, Theorem, (2)] should read as $\lambda^i_{jk} = \lambda^j_{ik} = \lambda^k_{ij}$.

of handles'. Analogously to [Wa66, Theorem 4], [Sk06'] $H\lambda_\pm[f] = f_\pm f_\mp^2$ and $6r_\pm[f] = f_\pm^3$. There is 'sliding handles' diffeomorphism $M_f \to M_g$. Under this diffeomorphism g_+, g_- go to $f_+, f_+ + f_-$. Since $r_+[f] = 0$, we obtain the required relations.[3] □

Remark 5.2. Let $D : \ker r_+ \to E^m(S^q_1 \sqcup S^q_2 \sqcup S^q_3)$ be the 'doubling' of the first component. The map D is well-defined and is a homomorphism [Sk11, Lemmas 3.1–3.3]. Clearly,

$$\lambda_{12}D = \lambda_{21}D = 0, \quad \lambda_{13}D = \lambda_{23}D = \lambda_+, \quad \text{and} \quad \lambda_{31}D = \lambda_{32}D = \lambda_-.$$

Clearly, $\sigma = \#_{23}D$. Hence for $m = 6k$, $q = 4k-1$ and $k > 1$ by Theorems 2.5.c and 2.6.b we have

$$H\lambda_+ + 2H\lambda_- = H\lambda_+\sigma = H\lambda_+\#_{23}D = \pm 2\mu D + H(\lambda_{12} + \lambda_{13})D = \pm 2\mu D + H\lambda_+.$$

Hence $\mu D = \pm H\lambda_-$.

Proof of Corollary 2.7. We have $\lambda_+ s_\pm = 0$. So for the first statement it suffices to prove that $H\lambda_+(\sigma\zeta - \zeta)w \neq 0$. This follows because

$$H\lambda_+\zeta w \overset{(1)}{=} Hw = 2 \neq 6 = 3Hw \overset{(4)}{=} (H\lambda_+ + 2H\lambda_-)\zeta w \overset{(5)}{=} H\lambda_+\sigma\zeta w, \quad \text{where}$$

- (1) follows because $\lambda_+\zeta = \mathrm{id}\,\pi_q(S^{m-q-1})$;
- (4) follows because $\lambda_+\zeta = \mathrm{id}\,\pi_q(S^{m-q-1})$ and by Lemma 2.4;
- (5) follows by Theorem 2.6.b.

The statement 'in other words' follows from the first statement by [Ha66C, Theorem 2.4], see also [Sk16h, Theorem 7.1]. □

Let $\Sigma : \pi_{4k-1}(S^{2k}) \to \pi_{4k}(S^{2k+1})$ the suspension homomorphism.

Theorem 5.3 (Haefliger)**.** *The following map is a monomorphism:*

$$\lambda_+ \oplus \lambda_- \oplus r_+ \oplus r_- : E^{6k}(T^{0,4k-1}) \to \pi_{4k-1}(S^{2k}) \oplus \pi_{4k-1}(S^{2k}) \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

Its image is the set of quadruples (a_+, a_-, b_+, b_-) such that $\Sigma(a_+ + a_-) = 0$. [Ha62l, Theorem in §6]*,* [Ha66C, Theorem 2.4 and appendix]*,* [Sk16h, Remark 7.2.b]

Theorem 5.4. *The kernel* $\ker H$ *is finite. The kernel* $\ker\Sigma$ *is generated by an element w such that $Hw = 2$* [Po85, Lecture 6, (7)].

Proof of Proposition 2.1. Part (a) holds by (c) and Theorem 5.4.

Part (b) holds by (d).

Parts (c), (d) hold by Theorems 5.3 and 5.4, because H is surjective for $k = 1, 3, 7$, and $H\pi_{4k-1}(S^{2k}) = 2\mathbb{Z}$ otherwise. □

Alternative proof of $H\lambda_-\omega = \pm 2$ for l even, using $\#\omega = 1$. By the Haefliger Theorem 5.3 $\Sigma\lambda_-\omega = -\Sigma\lambda_+\omega = 0$. Then by Theorem 5.4 $H\lambda_-\omega$ is an even number, say $2s$. Since $\lambda_+\omega = 0$ and $r_\pm\omega = 0$, by Theorems 5.3 and 5.4 we have that ω is divisible by s. Then $\#\omega$ is divisible by s. Since $\#\omega = 1$, we obtain $|s| = 1$. □

For $m = 6k$ and $q = 4k-1$ let (see the text after Connected Sum Theorem 2.2)

$$h_{+,-} := \frac{1}{2}H(\lambda_+ + \lambda_-), \quad E = E^m(S^q) \cong \mathbb{Z}, \quad \text{and} \quad \pi = \pi_q(S^{m-q-1}).$$

The map $\tau : \pi \to E^m(D^1 \times S^q)$ was essentially constructed in the construction of ζ.

[3] Sketches of alternative proofs of Theorems 2.2 and 2.6.b would work for any k if one proves higher-dimensional analogue of [Wa66, §4].

Lemma 5.5. *(a) We have* $\Sigma\lambda_+\sigma = \Sigma\lambda_+$.

(b) On K *we have* $h_{+,-}\sigma = 3h_{+,-}$.

(c) We have $h_{+,-}r\tau = 0$.

(d) We have $h_{+,-}\psi_-r\tau = H$.

(e) We have $E^m(D^1 \times S^q)/\operatorname{im}\tau \cong E \cong \mathbb{Z}$.

Proof. (a) We have $\Sigma\lambda_+\sigma = -\Sigma\lambda_-\sigma = -\Sigma\lambda_- = \Sigma\lambda_+$ by Theorem 2.6.a and the Haefliger Theorem 5.3.

(b) On K we have $2h_{+,-}\sigma = 2H\lambda_- + H\lambda_- = 6h_{+,-}$ by Theorem 2.6.

(c) We have

$$2h_{+,-}r\tau \overset{(1)}{=} 2h_{+,-}\psi_-\zeta \overset{(2)}{=} H(\lambda_+ - \lambda_-)\zeta \overset{(3)}{=} H - H = 0, \quad \text{where}$$

• (1) holds because $r\tau = \psi_-\zeta$;

• (2) holds because $\lambda_-\psi_- = -\lambda_-$ and $H\lambda_+\psi_- = H\lambda_+$ analogously to Symmetry Lemma 5.1.b;

• (3) holds because $\lambda_+\zeta = \operatorname{id}\pi$ and by Lemma 2.4.

(d) We have $h_{+,-}\psi_-r\tau = h_{+,-}\zeta = \frac{1}{2}(H + H) = H$ analogously to (d).

(e) This holds by [Sk11, Theorem 2.5], [Sk15, Theorem 1.7]. ☐

Proof of Corollary 2.8.b. Consider the following diagram:

$$E^m(D^1 \times S^q) \oplus K \oplus E \overset{r\oplus\sigma|_K\oplus s_-}{\to} E^m(T^{0,q}) \overset{r_+\oplus h_{+,-}}{\to} E \oplus \mathbb{Z}.$$

The map $r_+ \oplus h_{+,-}$ is surjective by Propositions 2.1.a,b. We have $r_+s_- = 0$ and $\lambda_+s_- = \lambda_-s_- = 0$, so $h_{+,-}s_- = 0$. Hence it suffices to prove that $(r_+ \oplus h_{+,-})(r \oplus \sigma|_K)$ is not surjective.

We have $r_+\tau = r_+\sigma = 0$. So by Lemma 5.5.b,c $(r_+ \oplus h_{+,-})(r \oplus \sigma)(\operatorname{im}\tau \oplus K) \subset 0 \oplus 3\mathbb{Z}$. Then $(r_+ \oplus h_{+,-})(r \oplus \sigma|_K)$ is not surjective by Lemma 5.5.e and the following simple result (applied to $A = E^m(D^1 \times S^q) \oplus K$ and $G = \operatorname{im}\tau \oplus K$).

If $G \subset A$ *is a subgroup of an abelian group* A *such that* $A/G \cong \mathbb{Z}$, *and* $\varphi : G \to \mathbb{Z}$ *is a non-surjective homomorphism, then no extension* $A \to \mathbb{Z} \oplus \mathbb{Z}$ *of the sum of* φ *and the zero map is surjective.* ☐

Proof of Corollary 2.8.a. (This proof corrects a mistake in [Sk15, proof of Theorem 2.8.a]. The map r of [Sk15, Theorem 2.8.a] is the map ψ_-r of this paper.) Consider the following diagram:

$$\begin{array}{ccc} E^m(D^1 \times S^q) \oplus K \oplus E & \xrightarrow{\psi_-r\oplus\sigma|_K\oplus s_-} & E^m(T^{0,q}) \\ \Big\downarrow{\scriptstyle (r_+r\oplus\lambda_+r)\oplus h_{+,-}\oplus\operatorname{id}E} & & \Big\downarrow{\scriptstyle r_+\oplus\lambda_+\oplus h_{+,-}\oplus r_-} \\ (E \oplus \pi) \oplus \mathbb{Z} \oplus E & \xrightarrow[\varphi]{} & E \oplus \pi \oplus \mathbb{Z} \oplus E \end{array}, \qquad \text{where}$$

$$\varphi(x, y, z, t) \ := \ (x,\ y + 2zw + \theta_1x,\ 3z + Hy + \theta_2x,\ t + x \pm z)$$

for some homomorphisms $\theta_1 : E \to \pi$ and $\theta_2 : E \to \mathbb{Z}$. Here $\pm$ is the sign depending only on k, the same as in Connected Sum Theorem 2.2.

The vertical arrows are isomorphisms by [Ha66C], see explanation in the fourth paragraph ($p = 0$) of [Sk15, Remark 1.8.a].[4]

[4]The inverse of the right vertical map from the diagram is $s_+ \oplus \zeta \oplus (z \mapsto z\omega) \oplus s_-$.

In this paragraph we prove that φ is an isomorphism. For this, it suffices to prove that the self-map $A = \begin{pmatrix} \mathrm{id}\,\pi & H \\ 2w & 3 \end{pmatrix}$ of $\pi \oplus \mathbb{Z}$ is an isomorphism. By Theorem 5.4 the group π is the sum of $\mathbb{Z}$ and a finite group. The map A maps the torsion subgroup to itself isomorphically. The determinant of A on the free part is $3 - 2Hw = -1$. Hence A is an isomorphism.

Thus it suffices to prove that the diagram is commutative (for some θ_1, θ_2).

The compositions of the diagram map every $t \in E$ to $(0,0,0,t)$ because $r_+s_- = \lambda_\pm s_- = 0$, and $r_-s_-t = t$.

The compositions of the diagram map every $u = h_{+,-}^{-1}z \in K$ to $(0, 2zw, 3z, \pm z)$ because

• $r_+\sigma = 0$;

• $\Sigma\lambda_+\sigma = \Sigma\lambda_+ = 0$ on K by Lemma 5.5.a, so $\lambda_+\sigma u = \frac{1}{2}(H\lambda_+\sigma u)w = (H\lambda_- u)w = (2h_{+,-}u)w = 2zw$ by Theorems 2.6.b and 5.4;

• $h_{+,-}\sigma = 3h_{+,-}$ on K by Lemma 5.5.b; and

• $r_-\sigma = \# = \pm h_{+,-}$ on K by Theorem 2.6.a and Connected Sum Theorem 2.2.

Since $\lambda_+ r\tau = \mathrm{id}\,\pi$ and $r_+r\tau = 0$, by Lemma 5.5.e there is a (non-canonical) homomorphism $\alpha : E \to E^m(D^1 \times S^q)$ such that $\alpha \oplus \tau$ is an inverse of $r_+r \oplus \lambda_+ r$. The compositions of the diagram map every $\alpha x + \tau y \in E^m(D^1 \times S^q)$ to $(x, \theta_1 x + y, \theta_2 x + Hy, x)$, where $\theta_1 := \lambda_+\psi_- r\alpha$ and $\theta_2 := h_{+,-}\psi_- r\alpha$, because

• $r_+\psi_- r = r_+ r = r_-\psi_- r$ because the orientations of the components used for $\psi_- r$ are the product orientations (as opposed to the boundary orientations);

• $\lambda_+\psi_- r\tau = \lambda_+\zeta = \mathrm{id}\,\pi$, so $\lambda_+\psi_- r(\alpha x + \tau y) = \theta_1 x + y$; and

• $h_{+,-}\psi_- r(\alpha x + \tau y) = \theta_2 x + Hy$ by Lemma 5.5.d. □

Remark 5.6 (low-dimensional versions of # and σ)**.** The definition of ζ after Connected Sum Theorem 2.2 works for $m = q+2 = 3$ and gives the 'standard link $\zeta(u) : S^1 \sqcup S^1 \to S^3$ of linking number u'. Let us give a less formal repetition of that construction for $m = q+2 = 3$. Let $\zeta(0)$ is be the trivial link. Now assume that $u \neq 0$. The first component of $\zeta(u)$ is the standard $S^1 = \mathrm{i}_{3,1}(0 \times S^1)$ in S^3. The second component of $\zeta(u)$ is contained in $\mathrm{i}_{3,1}(\partial D^2 \times S^1)$, makes $|u|$ turns around $S^1 \times 0$ and is oriented 'parallel' to $\mathrm{i}_{3,1}(0 \times S^1)$ when $u > 0$ and 'opposite' to $\mathrm{i}\,|_{S^1\times 0}$ when $u < 0$.

Define the link $\#\zeta(u)$ as in the definition of #, taking for connected summation a band close to 'standardly twisted' rectangle. The isotopy class of $\#\zeta(u)$ is independent of the choice of such a band (as opposed to PL band). It would be interesting to know which knot is $\#\zeta(u)$ (depending on u).

Define the link $\sigma\zeta(u)$ as in the definition of σ, taking for connected summation a band close to 'standardly twisted' rectangle. The isotopy class of $\sigma\zeta(u)$ is independent of the choice of such a band (as opposed to PL band). I conjecture that $\sigma\zeta(u)$ is not isotopic to $\zeta(u)$, at least for $|u| > 1$. Perhaps this conjecture could be (dis)proved using calculation of some invariant σ [Me20, Theorems 3 and 4.1].

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