# HOMOTOPY TRIANGULATIONS AND TOPOLOGICAL MANIFOLDS

John Hollingsworth and John Morgan

In [8] and [9] Kirby and Siebenmann answer the question: "Does a topological manifold have a triangulation?" Their answer is that, provided the dimension of M is not 4, there is one obstruction, $k_M \in H^4(M; \mathbb{Z}/2)$, which is the obstruction to lifting the stable topological tangent bundle of M to a PL bundle. Two related questions are 1.) "When can a closed topological manifold be triangulated up to homotopy type?" (i.e. "When does there exist $Q^n$ a PL manifold homotopy equivalent to $M^n$?") and 2.) "If M is a closed PL manifold what is the relation of the set of topological manifolds homotopy equivalent to M to the set of PL manifolds homotopy equivalent to M?" In this paper we will examine both questions and give partial answers to each.

Some things were known about the first question. For instance, Hsiang and Shaneson [7] show that if some fake k-torus, $\tau^k$, $(k \geqq 5)$ is homeomorphic to $T^k$ then there is a closed topological $(k+1)^-$ manifold which does not have the homotopy type of a closed PL manifold ($T^k = S^1 x \ldots x S^1$, k times, a fake torus, $\tau^k$, is a PL manifold homotopy equivalent to $T^k$ but not PL homeomorphic.) Kirby and Siebenmann, however, show that for $k \geq 5$, any fake torus, $\tau^k$, is homeomorphic to $T^k$. In dimension 5 another example is provided by Siebenmann, [15]. Let $W^4$ be the $\pi$-manifold of index 8 obtained by plumbing together 8 copies of the tangent disk bundle of $S^2$ (the Milnor manifold in

Mathematics Dept.
Princeton Univ.
1971

dimension 4, [3] section 5) $\partial W^4$ is a homology three sphere (the dodecahediel space) and Siebenmann shows that there is a closed topological manifold homotopy equivalent to $(W^4 \cup c(\partial W)) \times S^1$, but no such PL manifold exists. The following theorem (with $z=0$) provides the answer in terms of the cohomology ring of M.

<u>Theorem:</u> Let $M^n$ be a closed topological n-manifold, $n \neq 4$. Then there is a subset $K \hookrightarrow H^4(M;\mathbb{Z}/2)$ such that $k_M - z \in K$ if and only if there is a closed manifold $Q^n$ and a homotopy equivalence $f: M \longrightarrow Q$ with $f^* k_M = z$. If $H \hookrightarrow H^4(M;\mathbb{Z}/2)$ is the subgroup of elements of the form $Sq^2 x + (y)_2$ for $x \in H^2(M;\mathbb{Z}/2)$ and $y \in H^4(M;\mathbb{Z})$ then $K \hookrightarrow H$. Furthermore

1.) if M is 1 - connected then $K = H$

2.) if $\pi_1(M) \cong \mathbb{Z}^k = \mathbb{Z}\oplus\ldots\oplus\mathbb{Z}$, k times, and $n \geqq k+5$ then $K = H$

3.) if $\pi_1(M) = \mathbb{Z} * \ldots * \mathbb{Z}$, k times, and $n \geqq 6$ then $K = H$

4.) if $\pi_1(M) = \mathbb{Z}/q$ and $n \geq 5$ then $K = H$.

5.) if $M = N^4 \times S^1$ for some closed 1 - connected four manifold, N, then $K = \text{image } (Sq^2 : H^2(M;\mathbb{Z}/2) \longrightarrow H^4(M;\mathbb{Z}/2))$

5) implies that there is a closed manifold homotopy equivalent to $\mathbb{C}P^2 \times S^1$ with no PL structure, but all manifolds homotopy equivalent to $S^2 \times S^2 \times S^1$ are PL.

The paper is organized along the following lines: In section one we use geometry and the Kirby-Siebenmann obstruction theory to construct a homeomorphism between simply connected PL manifolds which is not homotopic to a PL homeomorphism. This implies, by Sullivan's work, [14], that [ ,TOP/PL] $\longrightarrow$ [ ,G/PL] is a nontrivial map. We then use Sullivan's calculation of G/PL, [17], to determine the above map and the space G/TOP. In section two we calculate the map G/TOP $\longrightarrow$ B(TOP/PL) = K( $\mathbb{Z}/2$,4). In section three we translate this homotopy theory into bundle theoretic results. We then study the situation for the various fundamental groups mentioned in the theorem stated above. To do this we must invoke the calculations of $L_n(\pi)$ and descriptions of surgery obstructions of various people. In this section we also compare the PL structures on M to the topological structures on M. All manifolds, unless otherwise stated are closed, topological and oriented. The results in sections one and two were originally proved by Siebenmann using different techniques and are well known (or widely suspected), but the authors were unable to find proofs in print.

The second author would like to express his gratitude to Professor Dennis Sullivan for the patient explanations of his research and Professor William Browder for his stimulating comments and his interest while this work was being done.

## Section 1 - Computation of G/TOP.

In this section we will give a proof that G/TOP is the path component of the constant path in $\Omega^4 G/PL$. A crucial step in the proof is showing that $TOP/PL \longrightarrow G/PL$ is nontrivial. To do this we construct a homeomorphism $M \xrightarrow{h} Q$ between simply connected PL manifolds so that h is not homotopic to a PL homeomorphism. Let $S_{PL}(Q)$ denote the homotopy triangulation set of Q, i.e. equivalence classes of maps $h:M \longrightarrow Q$, h a homotopy equivalence, M, a PL manifold. $(M,h)$ is equivalent to $(M',h')$ if there is a homotopy commutative diagram

$$\begin{tikzcd} M \arrow[dr, "h"] \arrow[d, "I"', "\cong"] & \\ M' \arrow[r, "h'"'] & Q \end{tikzcd}$$

with I a PL isomorphism.

Thus any homeomorphism $h:M \longrightarrow Q$ which is not homotopic to a PL homeomorphism is a non zero element in $S_{PL}(Q)$. The Sullivan structure sequence, [17], provides a one to one map $S_{PL}(Q) \xrightarrow{\omega} [Q,G/PL]$. Thus $\omega(M,h) : Q \longrightarrow G/PL$ is nontrivial, but since h is a homeomorphian $Q(M,h)$ factors through $TOP/PL \xrightarrow{\rho} G/PL$. This shows $\rho$ is nontrivial.

Note. Sullivan has shown that $TOP/PL \longrightarrow G/PL \longrightarrow BPL$ is nontrivial by different methods.

Definitions and Notation. Let $Q^n$ be a compact, oriented PL n-manifold. We assume that $\partial Q$ is orientation preserving PL homeomorphic to k copies of some PL manifold $M^{n-1}$, i.e. $f:\partial Q^n \xrightarrow{\cong} M^{n-1} \amalg \ldots \amalg M^{n-1}$. Let W be the PL space obtained by gluing the copies of $M^{n-1}$ together by the identity maps between them. Such a space, $W^n$, is a closed, PL, $\mathbb{Z}/k$ manifold, and $M^{n-1} \longrightarrow W^n$ is its Bockstein, written $M = \delta W$. We will denote a $\mathbb{Z}/k$ manifold by $(W,\delta W)$. Given $(W,\delta W)$, we may cut it along $\delta W$ to get $\overline{W}^n$, the original PL manifold with boundary. A PL $\mathbb{Z}/k$ manifold with boundary is defined similarily, see [1]. If $(W_1^n,\delta W_1)$ and $(W_2^n,\delta W)$ are PL $\mathbb{Z}/k$ manifolds and $f:W_1 \longrightarrow W_2$ is a map, we say that f is a map of $\mathbb{Z}/k$ manifolds provided f is transverse regular to $\delta W_2$ and $f^{-1}(\delta W_2) = \delta W_1$. Hence a map $f:W_1^n \longrightarrow W_2^n$ of $\mathbb{Z}/k$ manifolds induces a map $\overline{f}:\overline{W}_1 \longrightarrow \overline{W}_2$. A $\mathbb{Z}/k$ manifold map $f:W_1 \longrightarrow W_2$ is a homotopy equivalence of $\mathbb{Z}/k$ manifolds if there exists a map $g:W_2 \longrightarrow W_1$ of $\mathbb{Z}/k$ manifolds such that fog and gof are homotopic to $IdW_2$ and $IdW_1$, respectively, (the homotopies are also maps of $\mathbb{Z}/k$ manifolds). Hence a homotopy equivalence $f:W_1 \longrightarrow W_2$ of $\mathbb{Z}/k$ manifolds induces a homotopy equivalence $\overline{f}:\overline{W}_1 \longrightarrow \overline{W}_2$. A neighborhood of $\delta W_1$ is homeomorphic to $\delta W_1 x c(k \text{ points})$. $f:W_1 \longrightarrow W_2$ is a PL homeomorphian of $\mathbb{Z}/k$ manifolds if and only if $\overline{f}:\overline{W}_1 \longrightarrow \overline{W}_2$ is.

Kirby and Siebenmann, [9], show that $S^3xT^2$ has precisely two concordance classes of PL structures. We write $(S^3xT^2)_\Sigma$ for the nonstandard structure.

<u>Lemma 1.1.</u> (Shaneson [15].). Let $H:M^6_{PL} \longrightarrow S^3xT^2xI$ be a normal map and suppose $\partial M = \partial{+}M \cup \partial{-}M$, with $\partial{+}M = (S^3xT^2)_\Sigma$ and $\partial{-}M = S^3xT^2$. Assume $H|\partial{+}M$ is a homotopy equivalence and $H|\partial{-}M$ is a PL homeomorphism. Let $h = HxId_{\mathbb{C}P^2} : Mx\mathbb{C}P^2 \longrightarrow S^3xT^2xIx\mathbb{C}P^2$. If we use Farrell's fibering theorem [5] along the boundary to get $N^7 \longrightarrow S^3xIx\mathbb{C}P^2$ by a homotopy equivanence, and then use relative transversality to get a cobordism Q

between the components of $N_1^7$ we have $I(Q)$ = Index of $Q = (2k+1)\cdot 8$.

Theorem 1.2. Let $W^4$, $M^6$ be PL Z/2-manifolds with $\delta W = S^3$ and $\delta M = (S^3 x T^2)_\Sigma$. Let $h: M^6 \longrightarrow W^4 \times T^2$ be a homotopy equivalence of $\mathbb{Z}/2$-manifolds such that $\bar{h}_*[\bar{M}] = [\overline{W x T^2}]$. Suppose h is normally cobordant to $h': M' \longrightarrow W x T^2$ so that $h'|\delta M': \delta M' \longrightarrow S^3 x T^2$ is a PL homeomorphism. Then if h' is transverse regular to $W^4$,

$I(\overline{h'^{-1}(W)}) - I(\bar{W}) = (2k+1)\cdot 16$.

Proof: Let $H: N^7 \longrightarrow W^4 x T^2 x I$ be the PL normal cobordism between (M,h) and (M',h'). Cutting along the coboundaries gives $\bar{H}: \bar{N} \longrightarrow \overline{W^4 x T^2 x I}$, $\partial\bar{N} = \bar{M} \cup - \bar{M}' \cup Q \cup Q$ where Q is a normal cobordism between $(\delta M, h|\delta M)$ and $(\delta M', h'|\delta M')$. Crossing with $\mathbb{C}P^2$ gives $\bar{H} x Id_{\mathbb{C}P^2}: \bar{N} x \mathbb{C}P^2 \longrightarrow \overline{W x T^2} x I x \mathbb{C}P^2$. Since $h: M \longrightarrow W x T^2$ is a homotopy equivalence of Z/2 manifolds, $\bar{h} x Id_{\mathbb{C}P^2}: \bar{M} x \mathbb{C}P^2 \longrightarrow \overline{W x T^2} x \mathbb{C}P^2$ is a homotopy equivalence. Hence fibering twice by Farrell [5] (first on the boundary, and then using the relative version of the theorem in the interior) gives a homotopy equivalence $\bar{M}_1^8 \longrightarrow \bar{W}^4 x \mathbb{C}P^2$ where $\partial \bar{M}_1^8$ is two copies of the same manifold V, and V is homotopy equivalent to $S^3 x \mathbb{C}P^2 \subset (S^3 x T^2)_\Sigma\, x \mathbb{C}P^2$.

Now $h': M' \longrightarrow W x T^2$ is transverse regular to W and $h'|\delta M' : \delta M' \longrightarrow S^3 x T^2$ is a PL homeomorphism. Hence we take $\bar{M}_1' = \bar{h}'^{-1}(\bar{W}) x \mathbb{C}P^2$, and to calculate $I(\overline{h'^{-1}(\bar{W})}) - I(\bar{W})$ it suffices to calculate $I(\bar{M}_1') - I(\bar{W} x \mathbb{C}P^2)$.

Consider $H|Q^6 x \mathbb{C}P^2 : Q^6 x \mathbb{C}P^2 \longrightarrow S^3 x T^2 x I x \mathbb{C}P^2$. This is a normal cobordism between $((S^3 x T^2)_\Sigma x \mathbb{C}P^2, h x Id_{\mathbb{C}P^2}|(S^3 x T^2)_\Sigma x \mathbb{C}P^2)$ and $((S^3 x T^2) x \mathbb{C}P^2$, PL homeomorphism). We have used the fibering theorem to put the boundary transverse regular to $S^3 x \dot{I} x \mathbb{C}P^2$. Thus the 8-dimensional manifold $N_1^8$ obtained by relative transversality will have index $(2k+1)\cdot 8$ by lemma 1.1. Now $\bar{M}_1 \cup N_1 \cup - \bar{M}_1' \cup N_1$ is the boundary of a 9-dimensional manifold (using relative transversality to make H transverse regular to $W \times \mathbb{C}P^2 \times I$). Hence $I(\overline{M}_1) + 2I(N_1) = I(\overline{M}_1')$. But $I(\overline{M}_1) = I(\overline{W} \times \mathbb{C}P^2) = I(\overline{W})$ because $\overline{M}' \longrightarrow \overline{W} \times \mathbb{C}P^2$ is a homotopy equivalence. Thus

$I(\overline{W}) - I(\overline{M}_1') = 2(2k+1) \cdot 8 = (2k+1)\,16$

<u>Corollary 1.3</u>. Let $h: M \longrightarrow W^4 \times T^2$ be a homotopy equivalence of Z/2 manifolds where $\delta W = S^3$, $\delta M = (S^3 \times T^2)_\Sigma$. Then h is not normally cobordant to a PL homeomorphism.

<u>Proof</u>: If h is normally cobordant to the PL homeomorphism $h': M' \longrightarrow W^4 \times T^2$, then $I(\overline{h'^{-1}(W)}) - I(\overline{W}) = 0$, which is not an odd multiple of 16.

<u>Lemma 1.4.</u> Suppose $M^n$ and $Q^n$ are PL manifolds without boundary, $n \geq 5$, $h: Q \longrightarrow M$ is a homeomorphism and $M_\circ \subset M$ is an open subset. Let $Q_\circ = h^{-1}(M_\circ)$. Then if $\theta(h)$ is the Kirby-Siebenmann obstruction to isotoping h to a PL homeomorphism, $i^*\theta(h) = \theta(h|Q_\circ)$, where $i: M_\circ \longrightarrow M$ is inclusion.

<u>Proof:</u> The obstruction $\theta(h)$ is defined as follows: Let $\tau_{PL}(M): M \longrightarrow B_{PL}$ and $\tau_{TOP}(M): M \longrightarrow B_{TOP}$ classify the stable PL and topological tangent bundles of M, respectively, so that $\tau_{TOP}(M)$ is the composition $M \xrightarrow{\tau_{PL}(M)} B_{PL} \longrightarrow B_{TOP}$. Then $Q \xrightarrow{h} M \xrightarrow{\tau_{TOP}(M)} B_{TOP}$ classifies the stable topological tangent bundle of Q, and is therefore homotopic to $\tau_{TOP}(Q)$. By lifting the homotopy to $B_{PL}$, we may assume that the diagram

$$\begin{tikzcd} Q \arrow[r, "\tau_{PL}(Q)"] \arrow[d, "h"'] \arrow[dr, "\tau_{TOP}(Q)"] & B_{PL} \arrow[d] \\ M \arrow[r, "\tau_{TOP}(M)"'] & B_{TOP} \end{tikzcd}$$

commutes. Then each of $Q \xrightarrow{\tau_{PL(Q)}} B_{PL}$ and $Q \xrightarrow{h} M \xrightarrow{\tau_{PL(M)}} B_{PL}$ is a lifting of $\tau_{TOP}(Q) : Q \longrightarrow B_{TOP}$, and hence differs by a map $Q \longrightarrow TOP/PL$. If $\alpha \in H^3(M;\mathbb{Z}_2)$ is the homotopy class of this map, then $\theta(h) = (h^{-1})^*\alpha$. Lemma 1.4 follows easily.

Proposition 1.5. Let $M^n$ and $Q^n$ be PL manifolds without boundary with $n \geqq 5$, and let $h: Q \longrightarrow M$ be a homeomorphism. Suppose there is a PL embedding $f: S^3 \times \mathbb{R}^{n-3} \longrightarrow M$ so that $< \theta(h), f_*[S^3] > \neq 0$. Then $S^3 \times T^2 \times \mathbb{R}^{n-3}$ inherits a PL structure $\Theta$ from $Q$ (via $(h\times Id_{T^2})^{-1} \circ (f\times Id_{T^2})$). The PL manifold $(S^3\times T^2 \times \mathbb{R}^{n-3})_\Theta$ is concordant to $(S^3 \times T^2)_\Sigma \times \mathbb{R}^{n-3}$.

Proof. The identity map $(S^3\times T^2\times \mathbb{R}^{n-3})_\Theta \longrightarrow S^3\times T^2 \times \mathbb{R}^{n-3}$ is a homeomorphism with non-zero Kirby-Siebenmann obstruction. Hence $\Theta$ is not the standard structure. The only remaining candidate is $\Sigma \times \Phi$ where $\Phi$ is the standard structure on $\mathbb{R}^{n-3}$.

Lemma 1.6. Suppose $M^n$ is a closed topological manifold, $n \geq 5$, $Q$ is a PL manifold and $f: M \longrightarrow \text{int } Q$ is an embedding with a topological normal bundle $\nu_f$ that admits a PL reduction. Then there is a unique (up to isotopy) PL structure $\Theta$ on $M$ for which there is an isotopy $H: Q\times I \longrightarrow Q$ such that

(i) $H_0 = Id_Q$

(ii) $H\circ f(M)$ is a PL submanifold of $Q$ with PL normal bundle $\nu_f$, and

(iii) $H$ is fixed outside a given neighborhood of $f(M)$ in $Q$.

Proof. By the classification theorem of [9] it suffices to show that $\tau(M)$ has precisely one PL reduction (say to $\tau_{PL}(M)$) such that $\tau_{PL}(M) \oplus \nu_f$ is the PL bundle $\tau(Q)|f(M)$. This is clear.

We now construct a homeomorphism which is not homotopic to a PL homeomorphism.

Let $M^n$ be a closed 2-connected PL manifold with $H_3(M;\mathbb{Z}) = \mathbb{Z}_2$, $n \geq 9$. There exists an embedding $S^3 \underset{\iota}{\subsetneq} M$ so that $\iota_*[S^3]$ is the non-zero element of $H_3(M;\mathbb{Z})$ and $2\cdot S^3$ is zero in $\Omega_3(M)$. Thus $S^3$ is the coboundary of some $\mathbb{Z}_2$ manifold $W^4 \subset M$. Let $h\colon Q^n_{PL} \longrightarrow M^n_{PL}$ be a homeomorphism with $\langle\theta(h),[S^3]_2\rangle \neq 0$. (Such a homeomorphism exists since $I_{TOP}(M) \longrightarrow H^3(M;\mathbb{Z}_2)$ is a bijection where $I_{TOP}(M)$ is the topological isotopy classes of PL structures on M.) We will show that $h \times Id_{T^2}\colon Q \times T^2 \longrightarrow M \times T^2$ is not homotopic to a PL homeomorphism. Thus $h\colon Q \longrightarrow M$ is a homeomorphism, and $[Q,h] \in S_{PL}(M)$ is non zero.

We may assume that $W^4 \times T^2 \subset M \times T^2$ with a PL normal bundle $\nu$. Let $h \times Id_{T^2} = g$.

$$\begin{array}{ccc}
Q\times T^2 & \longrightarrow & M\times T^2 \\
\uparrow & & \uparrow \\
g^{-1}\nu & \longrightarrow & \nu \\
\uparrow & & \uparrow \\
g^{-1}(W^4\times T^2) & \longrightarrow & W^4\times T^2
\end{array}$$

Now $g^{-1}(\nu)$ is an open subset of $Q \times T^2$, so it inherits a PL structure, and $\nu(g^{-1}(W\times T^2) \longrightarrow g^{-1}(\nu))$ is a topological bundle which admits a PL reduction (via g). Thus there is a unique PL structure $\Theta$ on $g^{-1}(W\times T^2)$ such that there exists an isotopy $H\colon g^{-1}(\nu)\times I \longrightarrow g^{-1}(\nu)$

with

i ) $H_0 = Id$

ii ) $H_1((WxT^2)_\Theta)$ is a PL submanifold of $g^{-1}(\nu)$, (with a new triangulation in the same isotopy class)

iii) H is fixed near $\infty$.

Now $H_1 \cup Id_{QxT^2 - g^{-1}\nu}: QxT^2 \longrightarrow QxT^2$ is a homeomorphism, and the composition

$$\begin{tikzcd} QxT^2 \arrow[r] \arrow[rr, bend right, "\bar{h}"'] & QxT^2 \arrow[r, "hxIdT^2"] & MxT^2 \end{tikzcd}$$

is isotopic to h.

Now $\bar{h}$ is PL transverse regular to $W^4xT^2$ with preimage $(W^4xT^2)_\Theta$. By proposition 1.5, $(WxT^2)_\Theta|\bar{h}^{-1}(S^3xT^2)$ is $(S^3xT^2)_\Sigma$. Hence by corollary 1.2, $((WxT^2)_\Theta, \bar{h}|(WxT^2)_\Theta)$ is not normally cobordant to a PL homeomorphism. If $\bar{h}$ were homotopic to a PL homeomorphism, then relative PL transverse regularity along $WxT^2 \subset MxT^2$ would give a normal cobordism from $(WxT^2)_\Theta \xrightarrow{\bar{h}} WxT^2$ to a PL homeomorphism. Since $hxId_{T^2}$ is homotopic to $\bar{h}$, h is not homotopic to a PL homeomorphism.

Sullivan shows (see [14], [18]) that the map TOP/PL $\longrightarrow$ G/PL factors through $K(\mathbb{Z}_2,3)$. Kirby and Siebenmann show [9] that TOP/PL = $K(\mathbb{Z}_2,3)$. By the argument at the begining of this section, the map TOP/PL $\longrightarrow$ G/PL is non-zero. Hence in the factorization

$$\begin{tikzcd} TOP/PL \arrow[r] \arrow[dr, "\varphi"'] & G/PL \\ & K(\mathbb{Z}_2,3) \arrow[u] \end{tikzcd}$$

the map $\varphi$ is a homotopy equivalence.

We have
$$\begin{array}{ccccc} TOP/PL & \longrightarrow & G/PL & \longrightarrow & G/TOP \\ \varphi\downarrow\cong & & \| & & \\ K(\mathbb{Z}/2,3) & \longrightarrow & G/PL & \xrightarrow{S} & \bar{\Omega}^4 G/PL \end{array}$$

This implies $\Omega\, G/TOP = \bar{\Omega}^5\, G/PL$. Thus $\pi_i(G/TOP) = P_i$, the mod 4 sequence $0, \mathbb{Z}/2, 0, \mathbb{Z}$. We wish to calculate the homotopy type of G/TOP and the map f. To do this we will use the argument of Sullivan which calculates G/PL together with the fact that one may do topological surgery in high dimensions.

Let $\Omega_*(X; \mathbb{Z}/n)$ be $\mathbb{Z}/n$-bordism classes of maps of $\mathbb{Z}/n$ manifolds into X. If $T_n = S^1 \cup_{xn} D^2$, then $\Omega_*(X; \mathbb{Z}/n) \cong \Omega_{*+1}(X \wedge T_n)$. Thus $\Omega_*(X; \mathbb{Z}/n)$ is the homotopy theoretic bordism with $\mathbb{Z}/n$ coefficients.

Sullivan's argument goes as follows; see [11]. There are homomorphisms $\Omega_{4*}(G/PL) \xrightarrow{\sigma} \mathbb{Z}$, $\Omega_{4*}(G/PL; \mathbb{Z}/n) \xrightarrow{\sigma_n} \mathbb{Z}/n$, and $\Omega_{4*+2}(G/PL; \mathbb{Z}/2) \xrightarrow{\sigma_2^{4k+2}} \mathbb{Z}/2$ $\sigma$, $\sigma_n$, $\sigma_2^{4k+2}$ are all defined the same way, namely by taking the surgery obstruction. Thus to $f\colon M^{4k} \longrightarrow G/PL$ there is associated a surgery problem

$$\begin{array}{ccc} \nu_{4k} & \xrightarrow{\tilde{g}} & \xi \\ \downarrow & & \downarrow \\ N^{4k} & \xrightarrow{g} & M^{4k} \end{array}.$$

$\sigma([M,f])$ is then defined to be $\dfrac{I(N)-I(M)}{8} \in \mathbb{Z}$. $\sigma_n$ is defined analogously except that $\sigma_n([M_n^{4k},f]) = \left[\dfrac{I(N)-I(M)}{8}\right]_n$. $\sigma_2^{4k+2}$ is the Kervaire invariant of the resulting normal map $N^{4k+2} \longrightarrow M^{4k+2}$. See [3] and [20]. These homomorphisms satisfy the following properties:

1.) $\sigma$, $\sigma_n$, $\sigma_2^{4k+2}$ are multiplicative with respect to the index. ($\alpha\colon \Omega_*(X;G) \longrightarrow G$ is multiplicative with respect to the index if $\alpha([M,f] \cdot N) = \alpha([M,f]) \cdot I(N)$ ).

2.) $\sigma$ and the $\sigma_n$ are compatable with the natural maps $\mathbb{Z} \longrightarrow \mathbb{Z}/n$ and $\mathbb{Z}/n\cdot k \longrightarrow \mathbb{Z}/n$.

Using these homomorphisms Sullivan [18] constructs maps

$$\begin{array}{ll} G/PL \xrightarrow{K+\mathcal{L}} \prod_{i \geq 1} K(\mathbb{Z}/2,4i-2) \times K(\mathbb{Z}_{(2)},4i) & \\ \quad \searrow^{\rho} & \\ \qquad BO(odd). & \end{array}$$

(See [19, section 2,] for the definition of the localization of a space at a set of primes) $\rho$ induces an isomorphism of $G/PL(odd) \longrightarrow BO(odd)$ and $K + \mathcal{L} : G/PL_{(2)} \longrightarrow \prod_{i \geq 1} K(\mathbb{Z}/2,4i-2) \times K(\mathbb{Z}_{(2)},4i)$ is an isomorphism on $\pi_i$ for $i \neq 4$ and multiplication by 2 on $\pi_4$. It then follows that $G/PL_{(2)} = K(Z/2,2) \times_{\delta Sq^2} K(\mathbb{Z}_{(2)},4) \times \prod_{i \geq 2} K(Z/2,4i-2) \quad K(\mathbb{Z}_{(2)},4i)$. Where $\delta Sq^2 \in H^5(K(Z/2,2);\mathbb{Z}_{(2)})$ is the integral Bockstein of $Sq^2$, the unique element of order 2 in the group which is isomorphic to $\mathbb{Z}/4$. See [18].

Since in dimensions greater than or equal to 5 topological surgery is possible, we have maps $\Omega_{4*}(G/TOP) \longrightarrow Z$, $\Omega_{4*}(G/TOP;Z/n) \longrightarrow \mathbb{Z}/n$ and $\Omega_{4*+2}(G/TOP;\mathbb{Z}/2) \longrightarrow \mathbb{Z}/2$ which satisfy the same properties. (To define these maps on the low dimensional bordism, first cross with $\mathbb{C}P^2$ to shift the dimensions up above 4.) Thus we have a map $G/TOP_{(2)} \longrightarrow \prod_{i \geq 1} K(Z/2,4i-2) \times K(\mathbb{Z}_{(2)},4i)$ and it is an isomorphism on $\pi_i$ for all i. Of course, $G/PL(odd) \longrightarrow G/TOP(odd)$ is a homotopy equivalence.

G/TOP is universal with respect to homorphisms as above. This means given X and maps $\Omega_{4*}(X) \xrightarrow{\rho} \mathbb{Z}$, $\Omega_{4*}(X;Z/n) \xrightarrow{\rho n} Z/n$ and $\Omega_{4*+2}(X;Z/2) \xrightarrow[2]{\rho 4k+2} Z/2$, the homomorphisms being compatable with multiplication by the index and compatable with the natural maps from $\mathbb{Z} \longrightarrow Z/n$ and $\mathbb{Z}/n\cdot m \longrightarrow \mathbb{Z}/m$, then there is a unique map $X \xrightarrow{f} G/TOP$

such that $f^*\sigma = \rho$, $f^*\sigma_n = \rho_n$, and $f^*\sigma_2^{4k+2} = \rho_2^{4k+2}$. The fact that the homomorphisms must be compatable with the index may be stated as follows.

1.) $\rho: \Omega_{4*}(X) \longrightarrow \mathbb{Z}$ must factor through $\Omega_{4*}(X) \otimes_{\Omega_{4*}(pt)} \mathbb{Z}$ where $\mathbb{Z}$ is a $\Omega_{4*}(pt)$ module via multiplication by the index

2.) $\rho n: \Omega_{4*}(X: \mathbb{Z}/n) \longrightarrow \mathbb{Z}/n$ must factor through

$$\Omega_{4*}(X; \mathbb{Z}/n) \otimes_{\Omega_{4*}(pt;\, \mathbb{Z}/n)} \mathbb{Z}/n$$

3.) $\rho^{4k+2}: \Omega_{4*+2}(X; \mathbb{Z}/2) \longrightarrow \mathbb{Z}/2$ must factor through $\Omega_{4*+2}(X; \mathbb{Z}/2) \otimes_{\Omega_{4*}(pt;\, \mathbb{Z}/2)} \mathbb{Z}/2$ . For a proof of this fact see [11]

The connection between the homomorphisms and the product structure at 2 is given by the surgery formulae of Sullivan, see [18]. There are unique class $K_{4i+2}$ and $\mathcal{L}_{4i}$ with the property that if $g: M^{4k} \longrightarrow G/TOP$ $\sigma(g) =$
$= \langle g^*\mathcal{L} \cup L(M),[M]\rangle$, and if $h: N^{4k+2} \longrightarrow G/TOP$, then
$\rho_2^{4k+2}(h) = \langle h^*K \cup V^2(N),[N]\rangle$. These classes $\mathcal{L}_4 + \mathcal{L}_8 + \ldots \in H^{4*}(G/TOP; \mathbb{Z}_{(2)})$ and $K_2+K_6+ \ldots \in H^{4*+2}(G/TOP; Z/2)$ induce a homotopy equivalence
$G/TOP_{(2)} \longrightarrow \prod_{i\geq 1} K(\mathbb{Z}_{(2)}, 4i) \times K(\mathbb{Z}/2, 4i-2)$.

This is the splitting at 2 of G/TOP into a product of Eilenberg-MacLane spaces.

## §2 [K,G/TOP] ⟶ [K,BTOP/PL]

In this section we compute the image of the map $[K,G/TOP] \longrightarrow [K,BTOP/PL] \cong H^4(K;\mathbb{Z}/2)$ for finite complexes K.

Suppose $f: K \longrightarrow BTOP/PL$ factors through G/TOP, say

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[dr, "g"'] & BTOP/PL \\ & G/TOP \arrow[u, "\varphi"'] \end{tikzcd}$$

Then $f^*\iota = g^*\varphi^*\iota$ where $\iota \in H^4(BTOP/PL;\mathbb{Z}/2)$ is the fundamental class. Hence we wish to identify $\varphi^*\iota$ in $H^4(G/TOP;\mathbb{Z}/2)$. By theorem 16.6.2c of [6] the sequence

$$\mathbb{Z}/2 = H^4(BTOP/PL;\mathbb{Z}/2) \xrightarrow{\varphi^*} H^4(G/TOP;\mathbb{Z}/2) \longrightarrow H^4(G/PL;\mathbb{Z}/2)$$

is exact, so if we find a non-zero element in $H^4(G/TOP;\mathbb{Z}/2)$ that goes to zero in $H^4(G/PL;\mathbb{Z}/2)$, this element is of necessity $\varphi^*\iota$.

Let

$$\begin{tikzcd} & \vdots \arrow[d] \\ & X_4 \arrow[d] \\ & X_3 \arrow[d] \\ G/PL \arrow[uur, "f_4"] \arrow[ur, "f_3"] \arrow[r, "f_2"] \arrow[dr, "f_1"] \arrow[ddr, "f_0"'] & X_2 \arrow[d] \\ & X_1 \arrow[d] \\ & X_0 \end{tikzcd} \qquad \begin{tikzcd} & \vdots \arrow[d] \\ & Y_4 \arrow[d] \\ & Y_3 \arrow[d] \\ G/TOP \arrow[uur, "g_4"] \arrow[ur, "g_3"] \arrow[r, "g_2"] \arrow[dr, "g_1"] \arrow[ddr, "g_0"'] & Y_2 \arrow[d] \\ & Y_1 \arrow[d] \\ & Y_0 \end{tikzcd}$$

be Postnikov systems for G/PL, G/TOP respectively. Hence

$X_1 = X_0 = pt$, $X_3 = X_2 = K(\mathbb{Z}/2,2)$, $Y_0 = Y_1 = pt$, $Y_3 = Y_2 = K(\mathbb{Z}/2,2)$.

From the discussion in section 1 it follows that the $(i+1)^{st}$ k-invariant for G/PL is zero if $i \neq 4k$, has odd order if $i = 4k > 4$, and $k^4 = \delta Sq^2 \iota_2$. $Y_4 = K(\mathbb{Z},4) \times K(\mathbb{Z}/2,2)$. Since G/TOP(2) is a product of Edenberg MacLane spaces.

Now the map G/PL $\longrightarrow$ G/TOP induces a map of Postnikov systems, so we have

$$\begin{tikzcd}
0 = H^4(K(\mathbb{Z}/2,2);\mathbb{Z}) \arrow[r] \arrow[d] & H^4(Y_4;\mathbb{Z}) \arrow[r, shift left, "\mu"] \arrow[d, "h^*_4"] & H^4(K(\mathbb{Z},4);\mathbb{Z}) \arrow[l, shift left, "\Pi_1^*"] \arrow[r] & H^5(K(\mathbb{Z}/2,2);\mathbb{Z}) \arrow[d, "\cong"] \\
0 = H^4(K(\mathbb{Z}/2,2);\mathbb{Z}) \arrow[r] & H^4(X_4;\mathbb{Z}) \arrow[r, "\nu"] & H^4(K(\mathbb{Z},4);\mathbb{Z}) \arrow[r, "\tau"] & H^5(K(\mathbb{Z}/2,2);\mathbb{Z})
\end{tikzcd}$$

where the rows are exact, and $\Pi_1 : Y_4 = K(\mathbb{Z},4) \times K(\mathbb{Z}/2,2) \longrightarrow K(\mathbb{Z},4)$ is projection on to the first factor.

Let $x = h_4^* \Pi_1^* \iota_4$. We wish to show that $\nu(x) = 2\iota_4$. Indeed, if $\alpha$ represents a generator of $\pi_4(K(\mathbb{Z},4))$, then $\beta$ is twice

$$\begin{tikzcd}
 & & G/PL \arrow[r, "h"] \arrow[d, "f_4"] & G/TOP \arrow[d, "g_4"] & \\
S^4 \arrow[urr] \arrow[r, "\alpha"'] \arrow[rrrr, bend right, "\beta"'] & K(\mathbb{Z},4) \arrow[r] & X_4 \arrow[r] & Y_4 \arrow[r, "\Pi_1"] & K(\mathbb{Z},4)
\end{tikzcd}$$

a generator of $\pi_4(K(\mathbb{Z},4))$. Thus $\beta^* \iota_4 = 2\, \alpha^* \iota_4$. Since $\alpha^* : H^4(K(\mathbb{Z},4):\mathbb{Z}) \longrightarrow H^4(S^4;\mathbb{Z})$ is an isomorphism, $\nu(x) = 2\iota_4$.

Suppose $x = 2\gamma$. Then $\nu(\gamma) = \iota_4$, so $k^5(G/PL) = \tau(\iota_4) = 0$. As $k^5(G/PL)$ is known to be non-zero, x reduced modulo 2 is non-zero. Hence $(x)_2$ is a non-zero element of the kernel of $H^4(X_4;\mathbb{Z}/2) \longrightarrow H^4(K(\mathbb{Z},4);\mathbb{Z}/2)$.

If we consider the same diagram with $\mathbb{Z}/2$ coefficients we see that the kernel is

$$\begin{array}{ccccccc} 0 \longrightarrow & H^4(K(\mathbb{Z}/2,2);\mathbb{Z}/2) & \longrightarrow & H^4(Y_4;\mathbb{Z}) & \longrightarrow & H^4(K(\mathbb{Z},4)\,\mathbb{Z}/2) \\ & \downarrow & & \downarrow f_4^* & & \downarrow \\ 0 \longrightarrow & H^4(K(\mathbb{Z}/2,2);\mathbb{Z}/2) & \longrightarrow & H^4(X_4;\mathbb{Z}/2) & \longrightarrow & H^4(K(\mathbb{Z},4);\mathbb{Z}/2) \end{array}$$

precisely $\mathbb{Z}/2$, and furthermore, if $Sq^2\iota_2$ is the non-zero element of $H^4(K(\mathbb{Z}/2,2);\mathbb{Z}/2)$, its image in $H^4(X_4;\mathbb{Z}/2)$, say $Sq^2k_2$, is the non-zero element of the kernel. Hence $(x)_2 + Sq^2k_2 = 0$ in $H^4(X_4;\mathbb{Z}/2)$, but $(\Pi_1{}^*\iota_4)_2 + Sq^2\iota_2 \neq 0$ in $H^4(Y_4;\mathbb{Z}/2)$.

Let $L_4 = g_4^*\,\Pi_1{}^*\iota_4 \in H^4(G/TOP;\mathbb{Z})$ and let $k_2 = g_2^*\iota_2 \in H^2(G/TOP;\mathbb{Z}/2)$. Then we have proved the following lemma:

Lemma 2.1. The kernel of the map $H^4(G/TOP;\mathbb{Z}/2) \longrightarrow H^4(G/PL;\mathbb{Z}/2)$ is $\mathbb{Z}/2$, and its non-zero element is $(L_4)_2 + Sq^2k_2$ where $L_4 \in H^4(G/TOP;\mathbb{Z})$ and $k_2 \in H^2(G/TOP;\mathbb{Z}/2)$ are the fundamental classes.

Theorem 2.2. Let $K$ be a finite complex and let $a: K \longrightarrow BTOP/PL = K(\mathbb{Z}/2,4)$. Then there is a map $T: K \longrightarrow G/TOP$ such that

$$\begin{tikzcd} K \arrow[r, "a"] \arrow[dr, "T"'] & BTOP/PL \\ & G/TOP \arrow[u, "\varphi"'] \end{tikzcd}$$

homotopy commutes if and only if $\delta a^*\iota = \delta Sq^2c$ for some $c \in H^2(K;\mathbb{Z}/2)$.

Proof: Suppose such a $T$ exists. Then $a^*\iota = T^*\varphi^*\iota =$ $= T^*((L_4)_2 + Sq^2y) = (T^*L_4)_2 + Sq^2\,T^*k_2$. Hence if we set $c = T^*k_2$, $a^*L - Sq^2c = (T^*L_4)_2$, so $\delta(a^*L - Sq^2c) = \delta(T^*L_4)_2 = 0$.

Now suppose $\delta a^*\iota = \delta Sq^2 c$, $c \in H^2(K;\mathbb{Z}/2)$. Then $a^*\iota - Sq^2 c$ is the mod 2 reduction of an integral class, say $d \in H^4(K;\mathbb{Z})$. Let $\alpha : K \longrightarrow K(\mathbb{Z},4)$ represent $d$ and let $\beta : K \longrightarrow K(\mathbb{Z}/2,2)$ represent $c$. This gives a map

$$(\alpha,\beta) : K \longrightarrow K(\mathbb{Z},4) \times K(\mathbb{Z}/2,2) = Y_4', \text{ and}$$

$$(\alpha,\beta)^* \left((\iota_4)_2 + Sq^2\iota_2\right) = a^*\iota.$$

Since all the non-zero k-invariants of G/TOP have odd order (and are primitive) and K is finite dimensional, some odd multiple of $(\alpha,\beta)$ lifts to a map $T : K \longrightarrow G/TOP$, and $T^*((L_4)_2 + Sq^2 k_2) = a^*\iota$. Hence

$$\begin{tikzcd}
K \arrow[r, "A"] \arrow[dr, "T"'] & B\,TOP/PL \\
 & G/TOP \arrow[u, "\varphi"']
\end{tikzcd}$$

homotopy commutes and the theorem is proved.

## Section 3. Geometric Results

Let $H \hookrightarrow H^4(X; \mathbb{Z}/2)$ be the subgroup im $(Sq^2 H^2(X; \mathbb{Z}/2))$ + im $r_2(H^4(X; \mathbb{Z}))$. If $\xi \longrightarrow X$ is a topological bundle, let $k(\xi)$ denote the Kirby-Siebenmann obstruction to lifting $\xi$ to a PL bundle. Let $k_M = k(\nu_M)$ for any topological manifold $M$. By [9] $k_M$ is the only obstruction to triangulating $M$ provided that the dimension of $M$ is not 4.

Theorem 3.1. Given $\alpha \epsilon H^4(M; \mathbb{Z}/2)$ there is a topological bundle $\xi$ over $M$ with $k(\xi) = \alpha$ and $[T(\xi)] \epsilon H_{N+n}(T(\xi); \mathbb{Z})$ spherical if and only if $k_M - \alpha \epsilon H$.

Proof: By Spivak's theorem [17], $[T(\xi)]$ is spherical if and only if $\xi$ is fiber homotopy equivalent to $\nu_M$. Assuming such a $\xi$ exists, $\nu_M - \xi: M \longrightarrow B_{TOP}$ factors through $G/TOP$. The obstruction to lifting $(\nu_M - \xi)$ to a PL bundle thus lies in the image $[M, G/TOP] \xrightarrow{\varphi} [M, B(TOP/PL)]$. Theorem 2.2 implies that the obstruction to lifting $(\nu_M - \xi)$ lies in $H$. By additivity, the obstruction to lifting $(\nu_N - \xi)$ is $k_M - k(\xi)$. Thus $(k_M - k(\xi)) \epsilon H$.

Conversely if $\alpha \epsilon H^4(M; \mathbb{Z}/2)$ with $(k_M - \alpha) \epsilon H$, there is a map $\rho: M \longrightarrow G/TOP$ so that $\varphi \circ \rho: M \longrightarrow B\ TOP/PL$ is $(k_M - \alpha)$.

Let $\bar{\xi}$ be the bundle classified by $(\rho: M \to G/TOP \to B_{TOP})$, and $\xi = \bar{\xi} \oplus \nu_M$. Then the obstruction to lifting $\xi$ to $B_{PL}$ is $\alpha$.

Corollary 3.2: If $f: N^n \longrightarrow M^n$ is a homotopy equivalence between topological manifolds, then $(k_N - f^* k_M) \epsilon H$.

Proof: $f^* k_M$ is the obstruction to lifting $f^* \nu_M$ to $B_{PL}$.

But $f^*\nu_M$ has a spherical representative for $[T(\nu_M)]$, since $f$ is a homotopy equivalence.

Corollary 3.3: Given $M^n$, $n \neq 4$, and $\alpha \in H^4(M^n; \mathbb{Z}/2)$, there is a degree 1 normal map $f: N^n \longrightarrow M^n$ with $k_N = f^*\alpha$ if and only if $(k_M - \alpha) \in H$.

Proof: Suppose there is such a degree one normal map $(\tilde{f}, f): (\nu_N, N) \longrightarrow (\xi, M)$. Since $[T(\nu_N)]$ is spherical and $T(\tilde{f})_*[T(\nu_N)] = [T(\xi)]$, $[T(\xi)]$ is spherical.

But $f^*(k(\xi)) = k(\nu_N) = f^*\alpha$ and $f^*$ is monic. So that $k(\xi) = \alpha$. Thus $(k_M - \alpha) \in H$. Conversely given $\alpha$ with $(k_M - \alpha) \in H$, there is a topological bundle $\begin{matrix}\xi\\ \downarrow\\ M^n\end{matrix}$ with $S^{N+n} \xrightarrow{f} T(\xi)$ degree $+1$. Use topological transversality $(n \neq 4)$ to put $f$ transverse regular to $M^n \subset T(\xi)$. Let

$$\begin{tikzcd} \nu_N \arrow[r, "\tilde{g}"] \arrow[d] & \xi \arrow[d] \\ N^n \arrow[r, "g"] & M^n \end{tikzcd}$$

be the resulting normal map. Then $g$ is a degree one normal map and $k(\nu_N) = g^*\xi = g^*\alpha$.

We wish to narrow the gap between corollary 3.2 and 3.3 by stating exactly when the normal map of corollary 3.3 may be assumed to be a homotopy equivalence. This of course is the classical surgery situation. However, we have the freedom to change the bundle $\xi$ by maps of $M$ into $G/PL$. The set of homotopy classes of maps, $[M^n, G/PL]$, operates on pairs $(\xi, t)$ where $\xi$ is a topological bundle over $M^n$ and $t: S^{N+n} \longrightarrow T(\xi)$ is a sphericalization of $[T(\xi)]$. If $[f] \in [M, G/PL]$ then

$[f]\circ(\xi,t) = (\bar{\xi},\bar{t})$ where $\bar{\xi} = \xi \oplus \psi\circ f$ ($\psi$: $G/PL \to B_{PL}$ is the natural map). Conversely the pairs $(\xi,t)$ and $(\bar{\xi},\bar{t})$ differ by an element in $[M^n,G/PL]$ if $\bar{\xi}-\xi$ may be reduced to a PL bundle. Define $S_{(\xi,t)}\colon [M^n,G/PL] \longrightarrow L_n(\pi_1(M))$ as follows. (See [22] section 3 for the definition of $L_n(\pi_1(M))$.) $S_{(\xi,t)}([f])$ is the surgery obstruction of the problem arising from putting $\bar{t}\colon S^{N+n} \longrightarrow T(\bar{\xi})$ transverse regular to $M^n$, where $(\bar{\xi},\bar{t}) = [f]\cdot(\xi,t)$

<u>Proposition 3.4</u>: Given $(\xi,t)$ with $k(\xi) = \alpha$, there is a homotopy equivalence $g\colon N^n \longrightarrow M^n$ with $g^{-1*}(k_N) = \alpha$ if and only if $0 \in \operatorname{im}(S_{(\xi,t)})$.

<u>Proof</u>: The existence of $t\colon S^{N+n} \longrightarrow T(\xi)$ implies $\alpha = k(\xi) \in H$. Suppose $0 \in S_{(\xi,t)}$. Let $f\colon M \longrightarrow G/PL$ be such that $S_{(\xi,t)}([f]) = 0$, and let $(\bar{\xi},\bar{t}) = [f]\cdot(\xi,t)$. Putting $\bar{t}$ transverse regular to $M \hookrightarrow T(\bar{\xi})$, we have a surgery problem with 0 obstruction. Thus we may assume $\bar{t}^{-1}(M) = N$ is homotopy equivalent to $M$. We have

$$\begin{array}{ccc} \nu_N & \xrightarrow{\bar{t}} & \bar{\xi} \\ \downarrow & & \downarrow \\ N & \xrightarrow[\simeq]{\bar{t}|} & M \end{array} .$$

Let $\bar{t}|N = g$. Thus $g^*(\bar{\xi}) = \nu_N$; so that $g^*k(\bar{\xi}) = k_N$. Equivalently $\alpha = k(\bar{\xi}) = (g^{-1})^*k_N$.

Conversely if there is a $g\colon N \longrightarrow M$ a homotopy equivalence with $g^{-1*}(k_N) = \alpha$, let $\bar{\xi} = (g^{-1})^*\nu_N$ and $\bar{t} = S^{N+n} \longrightarrow$

$T(\nu_N) \xrightarrow{\tilde{g}} T(\bar{\xi})$. $k(\xi - \bar{\xi}) = k(\xi) - k(\bar{\xi}) = \alpha - \alpha = 0$. Thus $\xi - \bar{\xi}$ may be reduced to a PL bundle. Thus $(\bar{\xi}, \bar{t}) = [f]\cdot(\xi, t)$ for some $f\colon M \longrightarrow G/PL$, and $S_{(\xi,t)}([f]) = 0$.

Specializing to $\alpha = 0$ we have

Corollary 3.5: Let $M^n$ be topological manifold, $n \neq 4$, then $M^n$ is h.e. to a PL manifold $\Leftrightarrow$ $k_M \in H$ and for any $\xi$ a PL bundle over $M$ with $t\colon S^{N+n} \longrightarrow T(\xi)$ degree 1 $S_{(\xi,t)}$ contains 0 in its image. (Since $k_M \in H$ there will be pairs $(\xi, t)$.)

Of course the second condition in both corollaries is unpleasant. We wish to characterize it in other ways and make explicit computations for certain groups where information is known about the Wall groups.

Let $K \subset H \subset H^4(M; \mathbf{Z}/2)$ be the subset of $x$ such that $k_M + x$ is realizable by a homotopy equivalence, i.e. all $x$ such that there is a homotopy equivalence $g\colon N^n \longrightarrow M^n$ with $(g^{-1})^* k_N = k_M + x$ or $g^* k_M - k_N = g^*(x)$. The next four propositions are devoted to giving conditions under which $K = H$. The technique of proof in all four is to show that an arbitrary topological normal map

$$\begin{array}{ccc} \nu_N & \xrightarrow{\tilde{g}} & \xi \\ \downarrow & & \downarrow \\ N & \xrightarrow{g} & M \end{array}$$

may be altered to another normal map,

$(\tilde{g}',g'): (\nu_N', N') \longrightarrow (\xi', M)$, with 0 surgery obstruction and such that $\xi'-\xi$ has the structure of a PL bundle. Once we establish this fact, given any $\alpha \epsilon k_M + H$, there is a $(\xi,t)$ with $k(\xi) = \alpha$. Putting $t: S^{N+n} \longrightarrow T(\xi)$ transverse regular gives a normal map

$$\begin{array}{ccc} \nu_N & \longrightarrow & \xi \\ \downarrow & & \downarrow \\ g: N & \longrightarrow & M \end{array} .$$

The existence of $\xi', g'$ with $\xi'-\xi$ reducible to a PL bundle means the normal map $(g',\tilde{g}')$ is associated to a pair $(\xi',t')$ which differs from $(\xi,t)$ by some map $f: M \longrightarrow G/PL$. Thus $S_{(\xi,t)}([f])$ is the surgery obstruction of $(g')$ which is 0. This shows 0 is in the image of $S_{(\xi,t)}$. Thus we have shown:

<u>Lemma 3.6</u>: If for some manifold $M$ any normal map into $M$
$\begin{array}{ccc} \nu_N & \longrightarrow & \xi \\ \downarrow & & \downarrow \\ g: N & \longrightarrow & M \end{array}$ may be altered to $\begin{array}{ccc} \nu_{N'} & \longrightarrow & \xi' \\ \downarrow & & \downarrow \\ N' & \xrightarrow{g'} & M \end{array}$ with $\xi'-\xi$ a PL bundle and surgery obstruction of $g'$ equal 0 then $K = H$.

<u>Proposition 3.7</u>. If $M^n$, $n \geq 5$ is simply connected then $K = H$.

<u>Proof</u>: Let $(\tilde{g},g): (\nu_N, N) \longrightarrow (\xi, M)$ be a normal map into $M$. $S(g)$ is the index or Kervaire invariant of $g$. Since $n \neq 4$ there is a normal map $(\tilde{h},h): (\nu_W, W^n) \longrightarrow (\zeta, S^n)$ with $S(h) = -S(g)$ and $\zeta$ a PL bundle. This follows from plumbing theory. (See [3].)

Form
$$\begin{tikzcd} \nu_{N\#}\nu_W \arrow[r, "\widetilde{g}\#\widetilde{h}"] \arrow[d] & \xi\#\zeta \arrow[d] \\ N\#W \arrow[r, "g\#h"] & M^n\#S^n=M^n \end{tikzcd} \qquad S(g\#h) = S(g) + S(h) = 0.$$

$\xi\#\zeta-\xi = d^*\zeta$ where $d\colon M^n \longrightarrow S^n$ is a degree one map. Thus $\xi\#\zeta-\xi$ has a PL reduction. The lemma now applies to prove the proposition.

Proposition 3.8: If $\pi_1(M^n) = \mathbb{Z}/k$, $n \geq 5$ and $k \geq 1$ is any integer then $K = H$ for $M$, i.e. all $\alpha\epsilon k_M + M$ are realizable by homotopy equivalences.

Proof: Case 1: $n = 2\ell$

Petrie shows, [12], that there is an onto map

$L_{2\ell}(\mathbb{Z}/k) \xrightarrow{A} F_{r_{k,\ell}}$, where $F_{r_{k,\ell}}$ is the free abelian group of rank
$$r_{k,\ell} = \begin{cases} \frac{k-1}{2} & k \text{ odd} \\ \frac{k}{2} & k \text{ even, } \ell \text{ even} \\ \frac{k-2}{2} & k \text{ even, } \ell \text{ odd.} \end{cases}$$

It follows easily from his work (see[11] that if $M^{2\ell}$ is a closed topological manifold with $\pi_1(M^{2\ell}) = \mathbb{Z}/k$ that $[M^{2\ell}, G/TOP] \xrightarrow{S} L_{2\ell}(\mathbb{Z}/k) \xrightarrow{A} F_{r_{k,\ell}}$ is 0. Wall shows that $\widetilde{L}_{2\ell}(\mathbb{Z}/k)$ = cokernel of the split monomorphism $L_{2\ell}(e) \to L_{2\ell}(\mathbb{Z}/k)$ is free abelian of rank $r_{k,\ell}$ [22]. Thus the image $S \hookrightarrow$ image $L^{2\ell}(e)$. This means the only surgery obstruction of a normal map onto $M^{2\ell}$ is the simply connected obstruction over the top cell. Thus we may alter the normal map, as in the simply connected case, over the top cell to produce another normal map with 0 obstruction so that the difference of the bundles is PL.

Case 2. $n = 2\ell+1$ and $k$ is odd. There is an argument due to William Browder [2] which shows $[L_k^{2\ell+1}, G/TOP] \longrightarrow L_{2\ell+1}(Z/k)$ is 0 if $L_k^{2\ell+1}$ is a lens space with fundamental group $\mathbb{Z}/k$. This argument uses only the facts that 1) $L_k^{2\ell-1} \hookrightarrow L_k^{2\ell+1}$ induces an isomorphism on $\pi_1$, 2) $\partial\nu_{L^{2\ell-1}} = \tilde{L}^{2\ell-1} \times S^1$, where $\tilde{L}^{2\ell-1}$ is the universal cover of $L^{2\ell-1}$, and 3) that $\pi_1(L^{2\ell+1} - \mathrm{int}\ \nu_{L^{2\ell-1}}) \cong \mathbb{Z}$. There is a result of Levine [10] which states given $M^{2\ell+1}$ with $\pi_1(M) = \mathbb{Z}/k$, any k, there is a $W^{2\ell-1} \underset{i}{\hookrightarrow} M^{2\ell+1}$ with $i_*$ an isomorphism on $\pi_1, \partial\nu_W = \tilde{W}\times S^1$ and $\pi_1(M^{2\ell+1} - \mathrm{int}\ \nu_W) = \mathbb{Z}$. Thus Browder's argument applies any time the fundamental group is $\mathbb{Z}/k$, k odd, to show that any normal map onto $M^{2\ell+1}$ has 0 surgery obstruction provided $\pi_1(M^{2\ell+1}) \cong \mathbb{Z}/k$ for k odd.

Case 3. $n = 2\ell+1$, k even. Wall in [21] generalizes Browder's argument to $L_k^{2\ell+1}$ for k even. If $\ell$ is even, the result is the same. If $\ell$ is odd and $g\colon N \longrightarrow L^{2\ell+1}$ is a normal map Wall shows $S(g) = 0$ provided that $\sigma_2^{4k+2}(g|L^{2\ell}) = 0$ where $L^{2\ell} \hookrightarrow L_k^{2\ell+1}$ is dual to $a \in H^1(L_k; \mathbb{Z}/2)$. Again applying Levine's theorem these remarks are true for any manifold with fundamental group $\mathbb{Z}/k$, k even. So in case $\ell$ is even the argument procedes as in the other cases. If $\ell$ is odd and k is even we may have to change the normal map g to one, g', with $\sigma_2^{4k+2}(g|L^{2\ell}) = 0$ to insure its surgery obstruction is 0.