

08/27, DC: Exotic spheres

$$X = S^n, n \geq 5$$

surgery exact sequence: "algebra" "manifolds" "topology"

$$\dots \rightarrow \mathcal{N}(S^n \times \mathbb{I}, \text{rel } \partial) \xrightarrow{\sigma_{n+1}} L_{n+1}(e) \xrightarrow{\omega_n} \mathcal{F}(S^n) \xrightarrow{\zeta_n} \mathcal{N}(S^n) \xrightarrow{\sigma_n} L_n(e)$$

$$\Theta_n := \mathcal{F}(S^n) = \{ \Sigma^n \xrightarrow{\cong} S^n \} \text{ equivalence:}$$

$$\begin{array}{ccc} \Sigma_0 & \xrightarrow{f_0} & S^n \\ \cong \downarrow \alpha & & \\ \Sigma_1 & \xrightarrow{f_1} & \end{array}$$

Proposition

- $\Sigma^n \cong S^n$ (homeo)
- $\Sigma^n \cong \mathcal{D}^n \cup_f \mathcal{D}^n$, $f: S^{n-1} \xrightarrow{\cong} S^{n-1}$
- $\Theta_n =$ oriented diffeomorphism classes of smooth structures on S^n

$$\mathcal{N}(S^n) = \left\{ \begin{array}{ccc} \mathcal{V}_n & \xrightarrow{F} & \xi \\ \downarrow & & \downarrow \\ \mathcal{M} & \rightarrow & S^n \end{array} \right\} / \text{normal bordism}$$

Rem.: If $\alpha: \xi \cong \zeta$,
 $(f, \bar{f}) \sim (f, \alpha \circ \bar{f})$

Proposition

For any Poincaré cplx. X with reducible SNF, the group $[X, G/O]$ acts freely and transitively on $\mathcal{N}(X)$.

Proof (X mfd.)

$$G/O \rightarrow BO \rightarrow BG \text{ homotopy fibration}$$

pullback of $EO/G/O$ $\begin{array}{ccc} E & \xrightarrow{t} & \mathbb{R}^k \\ \downarrow \pi & & \downarrow \\ X & = & X \end{array}$ cone on a fibre homotopy equivalence.

There is a section $s_0: X \rightarrow \underline{\mathbb{R}}^k$. Make $t: E \rightarrow \underline{\mathbb{R}}^k$ transverse to $X \times \{0\}$.

$$M := t^{-1}(X \times \{0\}).$$

$$\nu_M = \underbrace{\nu_{M \hookrightarrow E}}_{\underline{\mathbb{R}}^k} \oplus \nu_E$$

$$= \pi^* \nu_X \oplus \pi^*(-E)$$

$$\begin{array}{ccc} \nu_M & \xrightarrow{\pi|_{\nu_M}} & \nu_X \oplus (-E) \\ \downarrow & & \downarrow \\ M & \xrightarrow{\pi|_M} & X \end{array}$$

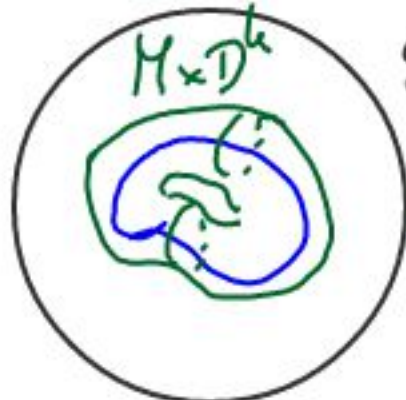
§ 2: The Pontryagin-Thom iso & the J-homomorphism

$$\Omega_n^k = \{ (M, \mathcal{J}) \mid \mathcal{J}: \nu_M \cong \underline{\mathbb{R}}^k \} / \text{framed, bordism}$$

Theorem

$$\Omega_n^k \cong \pi_n^S = \operatorname{colim}_{k \rightarrow \infty} \pi_{n+k}(S^k)$$

$c_{(M, \mathcal{J})}$:



$$M^n \longrightarrow T(\nu_M) \xrightarrow{pr} S^k$$

$$\begin{array}{c} M \times D^k \\ \parallel \\ M \times S^{k-1} \end{array}$$

$$[M, \mathcal{J}] \mapsto [c_{(M, \mathcal{J})}].$$

$$[f^{-1}(v), \mathcal{J}_f] \mapsto [f: S^{n+k} \rightarrow S^k]$$

Lemma

$$G := \operatorname{colim}_{k \rightarrow \infty} G(k+1) = \operatorname{colim}_{k \rightarrow \infty} \operatorname{Map}_{\pm 1}(S^k)$$

$$a) \text{ Ad: } \pi_n(G) \xrightarrow{\cong} \pi_n^S, [g] \mapsto [H(\text{Ad}(g))]$$

$$g: S^n \rightarrow \text{Map}_{\pm 1}(S^k, S^k)$$

$$\text{Ad}(g): S^n \times S^k \rightarrow S^k \quad (x, y) \mapsto g(x)(y)$$

$$\rightsquigarrow S^{n+k+1} \cong S^n * S^k \xrightarrow{H(\text{Ad}(g))} \Sigma S^k \cong S^{k+1}$$

$$b) \Omega^\infty S^\infty = \text{colim}_{k \rightarrow \infty} \Omega^k S^k = QS^\circ, \quad \pi_0(QS^\circ) = \mathbb{Z}$$

$$QS_i^\circ \cong QS_j^\circ \quad i, j \in \mathbb{Z}$$

$$\pi_n^S \cong \pi_n(QS_0^\circ) \cong \pi_n(QS_1^\circ) = \pi_n(SG).$$

Definition

$$O := \text{colim}_{k \rightarrow \infty} O(k)$$

$$F: \pi_n(O) \rightarrow \Omega_n^k, \quad I \mapsto (S^n, I)$$

Lemma

The following diagram commutes:

$$\begin{array}{ccc} \pi_n(O) & \xrightarrow{J_*} & \pi_n(G) \\ F \downarrow & & \downarrow \text{Ad} \\ \Omega_n^k & \xrightarrow{\text{P.-T.}} & \pi_n^S \end{array}$$

Theorem Bott

$n \pmod{8}$	0	1	2	3	4	5	6	7
$\pi_n(O)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	\mathbb{Z}	0	0	0	\mathbb{Z}

Theorem Serre

π_n^S is finite for $n \geq 1$.

Theorem

1) [Adams] $\gamma_n: \pi_n(O) \rightarrow \pi_n(G)$ is split injective
if $n \equiv 3 \pmod{4}$

2) [Quillen, Sullivan] $\text{Im}(\gamma_{4k-1})$ is a summand
of order $\text{Denom}(B_k/4k)$, B_k : k -th Bernoulli number
e.g. $|\text{Im}(\gamma_7)| = 240$.

Corollary

$$\pi_n(G/O) \cong \begin{cases} \text{coker}(\gamma_n) & n \neq 4k-1 \\ \text{coker}(\gamma_{4k}) \oplus \text{ker}(\gamma_{4k-1}) & \text{o/w} \end{cases}$$

Theorem Bott

$$P_k: \pi_{4k-1}(O) \rightarrow \mathbb{Z}$$

$$\varphi \mapsto \langle P_k E_\varphi, [S^{4k}] \rangle$$

k -th Pontryagin class $E_\varphi \rightarrow S^{4k}$ clutched by φ

$$\text{gen. } x \mapsto a_k (2k-1)!, \quad a_k = \begin{cases} 1 & k \text{ even} \\ 2 & k \text{ odd} \end{cases}$$

Theorem Kervaire-Milnor

$$bP_{n+1} := \text{Im}(\omega_{n+1}) \subseteq \Theta_n$$

equals $\{ \Sigma \mid \Sigma = \partial W, W \text{ parallelisable} \}$

There is an exact sequence

$$0 \rightarrow bP_{n+1} \rightarrow \Theta_n \rightarrow \text{coker}(\gamma_n) \xrightarrow{\kappa} \mathbb{Z}/2.$$

• $bP_{n+1} = 0$ if n is even.

• bP_{4k} is finite cyclic (but very large)

• $bP_{4k+2} \cong \begin{cases} 0 & 6, 14, 30, 62, \underline{126(?)} \\ \mathbb{Z}/2 & \text{o/w} \end{cases}$

$\Rightarrow \Theta_n$ is finite.