08/27, DC: Exotic spheres

$X = S^n$, $n \geq 5$

surgery exact sequence: "algebra" "manifolds" "topology"

$$\dots \to \mathcal{N}(S^n \times I, \mathrm{rel}\,\partial) \xrightarrow{\sigma_{n+1}} L_{n+1}(e) \xrightarrow{\omega_n} \mathcal{S}(S^n) \xrightarrow{\eta_n} \mathcal{N}(S^n) \xrightarrow{\sigma_n} L_n(e)$$

$\Theta_n := \mathcal{S}(S^n) = \{ \Sigma^n \xrightarrow{\simeq} S^n \}$ equivalence:

$$\begin{tikzcd} \Sigma_0 \arrow[dr, "f_0"] \arrow[d, "\cong"'] & \\ \Sigma_1 \arrow[r, "f_1"'] & S^n \end{tikzcd} \quad \circlearrowleft \simeq$$

<u>Proposition</u>

- $\Sigma^n \cong S^n$ (homeo)
- $\Sigma^n \cong D^n \cup_f D^n$, $f: S^{n-1} \xrightarrow{\cong} S^{n-1}$
- Θ_n = oriented diffeomorphism classes of smooth structures on S^n

$$\mathcal{N}(S^n) = \left\{ \begin{array}{ccc} \nu_M & \xrightarrow{\bar{f}} & \xi \\ \downarrow & & \downarrow \\ M & \longrightarrow & S^n \end{array} \right\} / \text{normal bordism}$$

Rem.: If $\alpha: \xi \cong \xi'$, $(f, \bar{f}) \sim (f, \alpha \circ \bar{f})$

<u>Proposition</u>

For any Poincaré cplx. X with reducible SNF, the group $[X, G/O]$ acts freely and transitively on $\mathcal{N}(X)$.

<u>Proof</u> (X mfd.)

$G/O \longrightarrow BO \longrightarrow BG$ homotopy fibration

pullback of $EO|_{G/O}$

$$\begin{array}{ccc} E & \xrightarrow{t} & \mathbb{R}^k \\ \downarrow \pi & & \downarrow \\ X & = & X \end{array}$$

cone on a fibre homotopy equivalence.

There is a section $s_0 : X \to \underline{\mathbb{R}}^k$. Make $t : E \to \underline{\mathbb{R}}^k$ transverse to $X \times \{0\}$.

$$M := t^{-1}(X \times \{0\}).$$

$$\nu_M = \underbrace{\nu_{M \hookrightarrow E}}_{\underline{\mathbb{R}}^k} \oplus \nu_E \qquad\qquad \begin{array}{ccc} \nu_M & \xrightarrow{\bar\pi|_M} & \nu_X \oplus (-E) \\ \downarrow & & \downarrow \\ M & \xrightarrow{\pi|_M} & X \end{array}$$

$$= \pi^* \nu_X \oplus \pi^*(-E)$$

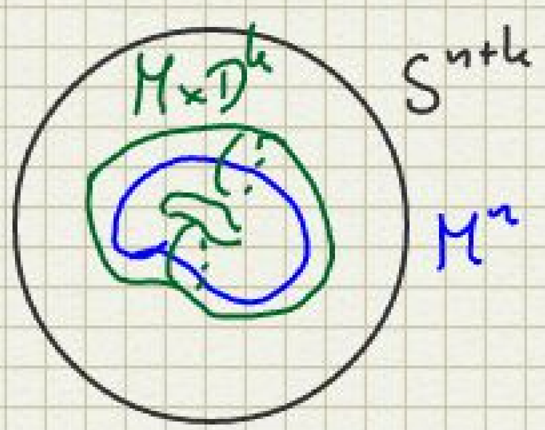
§2: The Pontryagin-Thom iso & the J-homomorphism

$$\Omega_n^{fr} = \{ (M^n, \vartheta) \mid \vartheta : \nu_M \cong \underline{\mathbb{R}}^k \} / \text{framed bordism}$$

<u>Theorem</u>

$$\Omega_n^{fr} \cong \pi_n^s = \operatorname*{colim}_{k \to \infty} \pi_{n+k}(S^k)$$

$c_{(M,\vartheta)}$:



$$\longrightarrow T(\nu_M) \xrightarrow{pr} S^k$$

$$T(\nu_M) = M \times D^k / M \times S^{k-1}$$

$$[M, \vartheta] \longmapsto [c_{(M,\vartheta)}].$$

$$[f^{-1}(*), \vartheta_f] \longmapsfrom [f : S^{n+k} \to S^k]$$

<u>Lemma</u>

$$G := \operatorname*{colim}_{k \to \infty} G(k+1) = \operatorname*{colim}_{k \to \infty} \mathrm{Map}_{\pm 1}(S^k)$$

a) $\mathrm{Ad} : \pi_n(G) \xrightarrow{\cong} \pi_n^s$, $[g] \mapsto [H(\mathrm{Ad}(g))]$

$$g: S^n \longrightarrow \mathrm{Map}_{\pm 1}(S^k, S^k)$$

$$\mathrm{Ad}(g): \quad S^n \times S^k \longrightarrow S^k \qquad (x,y) \longmapsto g(x)(y)$$

$$\rightsquigarrow \quad S^{n+k+1} \cong S^n * S^k \xrightarrow{H(\mathrm{Ad}(g))} \Sigma S^k \cong S^{k+1}$$

b) $\Omega^\infty S^\infty = \operatorname*{colim}_{k \to \infty} \Omega^k S^k = QS^0$, $\pi_0(QS^0) = \mathbb{Z}$

$$QS^0_i \simeq QS^0_j \qquad i,j \in \mathbb{Z}$$

$$\pi_n^S \cong \pi_n(QS^0) \cong \pi_n(QS^0_1) = \pi_n(SG).$$

<u>Definition</u>

$$O := \operatorname*{colim}_{k \to \infty} O(k)$$

$$F: \pi_n(O) \longrightarrow \Omega_n^{fr}, \quad \vartheta \mapsto (S^n, \vartheta)$$

<u>Lemma</u>

The following diagram commutes:

$$\begin{array}{ccc} \pi_n(O) & \xrightarrow{J_*} & \pi_n(G) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}} \\ \Omega_n^{fr} & \xrightarrow{\text{P.-T.}} & \pi_n^S \end{array}$$

<u>Theorem</u> Bott

$n \pmod 8$	0	1	2	3	4	5	6	7
$\pi_n(O)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

<u>Theorem</u> Serre

π_n^S is finite for $n \geq 1$.

Theorem

1) [Adams] $J_n: \pi_n(O) \longrightarrow \pi_n(G)$ is split injective if $n \equiv 3 \pmod 4$

2) [Quillen, Sullivan] $\mathrm{Im}(J_{4k-1})$ is a summand of order $\mathrm{Denom}(B_k/4k)$, B_k: k-th Bernoulli number

e.g. $|\mathrm{Im}(J_7)| = 240$.

Corollary

$$\pi_n(G/O) \cong \begin{cases} \mathrm{coker}(J_n) & n \neq 4k-1 \\ \mathrm{coker}(J_{4k}) \oplus \ker(J_{4k-1}) & \text{o/w} \end{cases}$$

Theorem Bott

$$\begin{aligned} p_k: \pi_{4k-1}(O) &\longrightarrow \mathbb{Z} \\ \varphi &\longmapsto \langle p_k E_\varphi, [S^{4k}] \rangle \end{aligned}$$

k-th Pontryagin class $\quad E_\varphi \to S^{4k}$ clutched by φ

gen. $x \longmapsto a_k (2k-1)!$, $a_k = \begin{cases} 1 & k \text{ even} \\ 2 & k \text{ odd} \end{cases}$

Theorem Kervaire-Milnor

$bP_{n+1} := \mathrm{Im}(\omega_{n+1}) \subseteq \Theta_n$

equals $\{ \Sigma \mid \Sigma = \partial W, W \text{ parallelisable} \}$

There is an exact sequence

$$0 \to bP_{n+1} \to \Theta_n \to \mathrm{coker}(J_n) \xrightarrow{K} \mathbb{Z}/2.$$

- $bP_{n+1} = 0$ if n is even.

- bP_{4k} is finite cyclic (but very large)
- $bP_{4k+2} \cong \begin{cases} 0 & 6, 14, 30, 62, \underline{126 (?)} \\ \mathbb{Z}/2 & o/w \end{cases}$

$\Longrightarrow \quad \Theta_n$ is finite.