Proposition 3.1 *The following relation holds in $U^2(X)$:*

$$u +_{H} v = F_U(u,v) = u + v + \sum_{k\geq 1,\, l\geq 1} \alpha_{kl}\, u^k v^l, \tag{1}$$

where the coefficients $\alpha_{kl} \in \Omega_U^{-2(k+l-1)}$ do not depend on X. The series $F_U(u,v)$ given by (1) is a formal group law over the ring $\Omega_U = \Omega_U^*$.

Proof. We first do calculations with the universal example $X = \mathbb{C}P^\infty \times \mathbb{C}P^\infty$. Then

$$U^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = \Omega_U^*[[\underline{u},\underline{v}]],$$

where $\underline{u},\underline{v}$ are canonical geometric cobordisms given by the projections of $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$ onto its factors. We therefore have the following relation in $U^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$:

$$\underline{u} +_{H} \underline{v} = \sum_{k,l\geq 0} \alpha_{kl}\, \underline{u}^k \underline{v}^l, \tag{2}$$

where $\alpha_{kl} \in \Omega_U^{-2(k+l-1)}$.

Now let the geometric cobordisms $u, v \in U^2(X)$ be given by maps $f_u, f_v \colon X \to \mathbb{C}P^\infty$ respectively. Then $u = (f_u \times f_v)^*(\underline{u})$, $v = (f_u \times f_v)^*(\underline{v})$ and $u +_{H} v = (f_u \times f_v)^*(\underline{u} +_{H} \underline{v})$, where $f_u \times f_v \colon X \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$. Applying the Ω_U^*-module map $(f_u \times f_v)^*$ to (2) we obtain the required formula (1). The fact that $F_U(u,v)$ is a formal group law follows directly from the properties of the group multiplication $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$. $\square$

Theorem 3.2 (Buchstaber)

$$F_U(u,v) = \frac{\sum_{i,j\geq 0}[H_{ij}]u^i v^j}{\left(\sum_{r\geq 0}[\mathbb{C}P^r]u^r\right)\left(\sum_{s\geq 0}[\mathbb{C}P^s]v^s\right)},$$

where H_{ij} $(0 \leq i \leq j)$ are Milnor hypersurfaces and $H_{ji} = H_{ij}$.

Proof. Set $X = \mathbb{C}P^i \times \mathbb{C}P^j$ in Proposition 3.1. Consider the *Poincaré–Atiyah duality* map $D\colon U^2(\mathbb{C}P^i \times \mathbb{C}P^j) \to U_{2(i+j)-2}(\mathbb{C}P^i \times \mathbb{C}P^j)$ and the map $\varepsilon\colon U_*(\mathbb{C}P^i \times \mathbb{C}P^j) \to U_*(pt) = \Omega_*^U$ induced by the projection $\mathbb{C}P^i \times \mathbb{C}P^j \to pt$. Then the composition

$$\varepsilon D\colon U^2(\mathbb{C}P^i \times \mathbb{C}P^j) \to \Omega^U_{2(i+j)-2}$$

takes geometric cobordisms to the bordism classes of the corresponding submanifolds. In particular, $\varepsilon D(u +_{H} v) = [H_{ij}]$, $\varepsilon D(u^k v^l) = [\mathbb{C}P^{i-k}][\mathbb{C}P^{j-l}]$. Applying εD to (1) we obtain

$$[H_{ij}] = \sum_{k,\,l} \alpha_{kl}[\mathbb{C}P^{i-k}][\mathbb{C}P^{j-l}].$$

Therefore,

$$\sum_{i,j}[H_{ij}]u^i v^j = \Bigl(\sum_{k,\,l}\alpha_{kl}u^k v^l\Bigr)\Bigl(\sum_{i\geq k}[\mathbb{C}P^{i-k}]u^{i-k}\Bigr)\Bigl(\sum_{j\geq l}[\mathbb{C}P^{j-l}]v^{j-l}\Bigr),$$

which implies the required formula. $\square$

Theorem 3.3 (Mishchenko). *The logarithm of the formal group law of geometric cobordisms is given by*

$$g_U(u) = u + \sum_{k\geq 1}\frac{[\mathbb{C}P^k]}{k+1}u^{k+1} \in \Omega_U \otimes \mathbb{Q}[[u]].$$

Proof. We have

$$dg_U(u) = \frac{du}{\left.\frac{\partial F_U(u,v)}{\partial v}\right|_{v=0}}.$$

Using the formula of Theorem 3.2 and the identity $H_{i0} = \mathbb{C}P^{i-1}$, we calculate

$$dg_U(u) = \frac{1 + \sum_{k>0}[\mathbb{C}P^k]u^k}{1 + \sum_{i>0}([H_{i1}] - [\mathbb{C}P^1][\mathbb{C}P^{i-1}])u^i}.$$

A calculation of Chern numbers shows that $[H_{i1}] = [\mathbb{C}P^1][\mathbb{C}P^{i-1}]$. Therefore, $dg_U(u) = 1 + \sum_{k>0}[\mathbb{C}P^k]u^k$, which implies the required formula. $\square$

Theorem 3.4 (Quillen). *The formal group law F_U of geometric cobordisms is universal.*

Proof. Let $\mathcal{F}$ be the universal formal group law over a ring A. Then there is a homomorphism $r\colon A \to \Omega_U$ which takes $\mathcal{F}$ to F_U. The series $\mathcal{F}$, viewed as a formal group law over the ring $A \otimes \mathbb{Q}$, has the universality properties for all formal group laws over $\mathbb{Q}$-algebras. Such a formal group law is determined by its logarithm, which is a series with leading term u. It follows that if we write the logarithm of $\mathcal{F}$ as $\sum b_k \frac{u^{k+1}}{k+1}$ then the ring $A \otimes \mathbb{Q}$ is the polynomial ring $\mathbb{Q}[b_1, b_2, \ldots]$. By Theorem 3.3, $r(b_k) = [\mathbb{C}P^k] \in \Omega_U$. Since $\Omega_U \otimes \mathbb{Q} \cong \mathbb{Q}[[\mathbb{C}P^1], [\mathbb{C}P^2], \ldots]$, this implies that $r \otimes \mathbb{Q}$ is an isomorphism.

By the Lazard Theorem the ring A does not have torsion, so r is a monomorphism. On the other hand, Theorem 3.2 implies that the image $r(A)$ contains the bordism classes $[H_{ij}] \in \Omega_U$, $0 \leq i \leq j$. Since these classes generate the whole ring Ω_U, the map r is onto and thus an isomorphism. $\square$