08/28, DC: Exotic spheres II

$$\ldots \to \mathcal{N}(S^n \times I, \mathrm{rel}\,\partial) \xrightarrow{\sigma_{n+1}} L_{n+1}(e) \xrightarrow{\omega} \mathcal{S}(S^n) \xrightarrow{\eta} \mathcal{N}(S^n) \xrightarrow{\sigma_n} L_n(e)$$

$$\mathcal{N}(S^n \times I, \mathrm{rel}\,\partial) \overset{(*)}{\cong} \pi_{n+1}(G/O), \qquad \mathcal{S}(S^n) \cong \Theta_n, \qquad \mathcal{N}(S^n) \cong \pi_n(G/O)$$

__Theorem__ Kervaire-Milnor

$$0 \to bP_{n+1} \to \Theta_{n+1} \to \mathrm{coker}(J_n) \xrightarrow{K} \mathbb{Z}/2$$

__Rem.:__ exact sequence of abelian groups

__Addendum:__ $\cdot\ \mathcal{N}(X, \mathrm{rel}\,\partial) \simeq [X/\partial X, G/O]$

$\cdot$ X closed: $\mathcal{N}(X - (\mathring{D}^n \sqcup \mathring{D}^n), \mathrm{rel}\, S^{n-1} \sqcup S^{n-1}) \simeq \mathcal{N}(X)$

$\Rightarrow$ (*) above

__Rem.:__ $\pi_n(G/O) \cong \Omega_n^{alm} = \{(M^n, \ell) \mid \ell : \nu_{\dot{M}} \cong \mathbb{R}^k\} /$ alm. framed bordism

[Notation: $\dot{M} \triangleq M$ with a small disc removed.]

__Goal:__ $\sigma : \Omega_{4k}^{alm} \xrightarrow{\sigma} 8\mathbb{Z}$

Find the minimal signature of an almost framed manifold.

$$\tau_{S^{n-1}} \nearrow \tau_{\dot{M}} \cong \underline{\mathbb{R}}^k, \qquad \tau_{S^{n-1}} \searrow \tau_{D^n} \cong \underline{\mathbb{R}}^k$$

compare framing $\rightsquigarrow f_{(M,\ell)} \in \pi_{n-1}(O)$

$J(\mathcal{F}(f_{(M,\vartheta)})) = 0 \qquad (\dot{M}, \vartheta)$ is null bordism.

Signature theorem [Hirzebruch]

M^{4k} closed, oriented smooth mfd.

- $\sigma(M) \in \mathbb{Z}$
- $\sigma(M) = \sigma(H^{2k}(M) \times H^{2k}(M) \longrightarrow \mathbb{Q})$
- $p(M) \in H^{4*}(M; \mathbb{Z})$

Theorem

There is a polynomial $L_k \in \mathbb{Q}[p_1, p_2, \ldots]$ s. th.

$$\sigma(M) = \langle L_k(p(TM)), [M] \rangle$$

e.g.: $L_1 = 1/3\, p_1, \quad L_2 = 1/45\,(7p_2 - p_1^2)$

$L_k = s_k p_k + \ldots, \quad s_k = \dfrac{2^{2k}(2^{2k-1} - 1) B_k}{(2k)!} \neq 0$

Corollary Kervaire-Milnor '63

$\Sigma^{4k} \in \Theta_{4k}$ is stably parallelisable.

Proof

$$\sigma(\Sigma^{4k}) = 0 = \langle s_k p_k(T\Sigma), [\Sigma] \rangle$$

$$\Rightarrow p_k(T\Sigma) = 0$$

$$\Rightarrow \tau_\Sigma \cong \underline{\mathbb{R}}^k \qquad p_k : \pi_{4k-1}(O) \rightarrowtail \mathbb{Z}$$

$$\pi_{4k}(G/O) \xrightarrow{\sigma} L_{4k}(e)$$

$$\sigma\begin{pmatrix} \nu_M & \xrightarrow{\bar{f}} & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & S^{4k} \end{pmatrix} = \sigma_M - \sigma_{S^{4k}} .$$

$$\sigma(f, \bar{f}) = \sigma_M = \langle -L(\nu_M), [M] \rangle$$

$$= \langle s_k p_k(\nu_M), [M] \rangle$$

$$\Rightarrow |bP_{4k}| = 8 \cdot a_k \cdot 2^{2k-2} \cdot \underbrace{(2^{2k-1} - 1)\,\mathrm{Num}(B_k/4k)}_{\text{odd}}$$

$$a_k = \begin{cases} 1 & k \text{ even} \\ 2 & k \text{ odd.} \end{cases}$$

$$\rightsquigarrow \quad bP_8 \cong \mathbb{Z}/28 \quad , \quad bP_{12} \cong \mathbb{Z}/992$$

<u>"Law of conservation of manifolds"</u>

$$\pi_{4k}(G/O) \longrightarrow \underset{\substack{\cong \\ 8\mathbb{Z}}}{L_{4k}(e)} \longrightarrow \mathcal{S}(S^{4k-1})$$

$$\pi_{4k+2}(G/O) \xrightarrow{K} \underset{\substack{\cong \\ \mathbb{Z}/2}}{L_{4k+2}(e)} \longrightarrow \mathcal{S}(S^{4k+1})$$

$$1 \longmapsto \Sigma_K$$

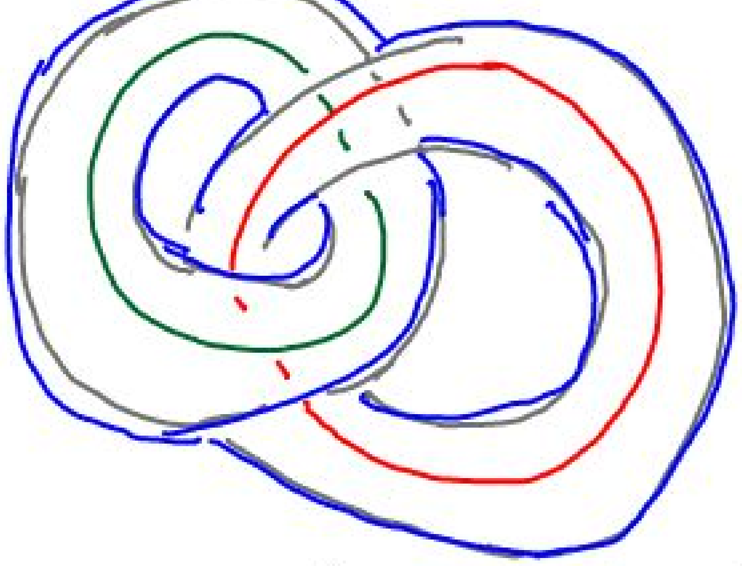
see below for definition

$$\Sigma_k \cong S^{4k+1} \iff K_{4k+2} \neq 0$$

Construction of exotic spheres (by Plumbing)

Plumbing: $D^{p+1} \tilde{\times}_\alpha S^{q+1}$, $D^{q+1} \tilde{\times}_\beta S^{p+1}$

$\alpha \in \pi_q(SO_{p+1})$, $\beta \in \pi_p(SO_{q+1})$



More generally, take a graph labelled by bundles. $\qquad p = q$.

$$W_M = W^{4k}(E_8, \tau_{S^{2k}}, \dots, \tau_{S^{2k}}) \quad , \quad \partial W_M = \Sigma_M^{4k-1}$$

$$\lambda_{W_M} = E_8 \qquad\qquad bP_{4k} \cong C([\Sigma_M]).$$

$$\Longrightarrow \sigma(\lambda_{W_M}) = 8$$

$$\begin{array}{ccccc} & & L_{4k}(e) & & \\ & & \downarrow & \searrow & \\ \to & & L_{4k}(\pi) & \to & \mathcal{S}(M^{4k-1}) \to \end{array}$$

$$[E_8] \cdot (N \xrightarrow{f} M) = [N \# \Sigma_M \longrightarrow M]$$

Rem.: $W_M \cup D^{4k}$ is a PL manifold with no smooth structure

$A: L_{4k+2}(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}/2$ (Arf invariant)

$\mathcal{A} := (H, \lambda, \mu) := \left(\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$

$W_k := W\left(\underset{\tau_{S^{2k+1}}}{\bullet} \!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!\underset{\tau_{S^{2k+1}}}{\bullet} \right) \qquad W_k \xrightarrow{\deg 1} D^{4k+2}$

$\tau_{S^{2k+1}} \cong \underline{\mathbb{R}}^{2k+1} \iff 2k+1 = 1, 3, 7$

$(H_{W_k}, \lambda_{W_k}, \mu_{W_k}) \cong \mathcal{A}.$

$\partial W_k =: \Sigma_k \in \Theta_{4k+1}.$

- Σ_k is non-standard if $4k+2 \neq 2, 6, 14, 30, 62, 126?$

<u>Theorem</u> Kervaire

$W_k^{10} \cup D^{10}$ admits no smooth structure.

<u>Inertia</u>

M^n: closed, oriented smooth mfd.

$\Theta_n \supseteq I(M) := \{\Sigma \mid M \# \Sigma \cong M\}$. <u>inertia group</u>

$I(M) \supseteq I_H(M) := \{\Sigma \mid f: M \# \Sigma \cong M,\ f \simeq \mathrm{id}\}$

<u>homotopy inertia group</u>

<u>Exercise:</u> $\Sigma \in I_H(M) \iff [M \# \Sigma \xrightarrow{\mathrm{id}} M] = [\mathrm{id}_M] \in \mathcal{S}(M).$

$$V = V_{2k+2,2} := S^{2k} \tilde{\times}_\tau S^{2k+1} \longrightarrow S^{2k+1}, \quad 2\tau_{S^{2k+1}} = 0$$

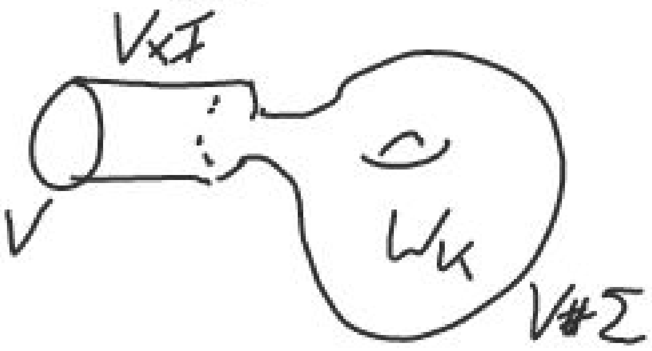
<u>Proposition</u>

1) $\Sigma_K \in I(V)$
2) $\Sigma_K \in I_H(V) \iff K_{4k+2} \neq 0$

<u>Proof of 1)</u>

$$V \# \Sigma = \underset{\displaystyle L_{4k+2}(e)}{\underset{\cap}{[\mathcal{A}]}} \cdot [V \xrightarrow{id} V]$$

<u>i.e.</u>



$$W := (V \times I) \natural W_K$$
$$\partial W = V \sqcup (V \# \Sigma_K)$$

$$H_{2k+1}(W) \cong H_{2k+1}(V) \oplus H_{2k+1}(W_K)$$
$$\mathbb{Z} \oplus \mathbb{Z}^2$$
$$\langle z \rangle \qquad \langle x, y \rangle$$

$\rightsquigarrow$ Do surgery on $z + x$.

$\Rightarrow$ h-cobordism from V to $V \# \Sigma$.

2) left as exercise.

<u>Exercise:</u> Show that if $K_{4k+2} \neq 0$, then there is $f : V \xrightarrow{\simeq} V$ not homotopic to a diffeomorphism.

(Notes taken by C.W.)