1. Manifold Atlas : Regensburg Surgery Blockseminar 2012

The surgery step (Nicolas Ginoux and Carolina Neira Jiménez)

Problem: *Given a normal map $(\overline{f}, f)$ from a closed smooth manifold M to a finite n-dimensional connected Poincaré complex X, can one change M and $(\overline{f}, f)$ - but neither ξ nor X - to a normal map $(\overline{f'}, f')$ from a new closed smooth manifold M' to X such that $f' : M' \to X$ is a homotopy equivalence?*

1.1. **Attaching cells.**

Definition 1.1. $f\colon Y \to X$ map between CW-complexes, $k \geq 1$

$$\pi_k(f) := \left[\begin{array}{ccc} S^{k-1} & \xrightarrow{q} & Y \\ {\scriptstyle j}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ D^k & \xrightarrow{Q} & X \end{array}\right]$$

Note:

- $k \geq 2$: group, abelian for $k \geq 3$
- $\cdots \pi_{l+1}(Y) \to \pi_{l+1}(X) \to \pi_{l+1}(f) \to \pi_l(Y) \cdots$
- $k \geq 2$: $\pi_1(Y)$-action on $\pi_k(f)$

Whitehead: $f\colon Y \to X$ is homotopy equivalence $\Leftrightarrow \pi_k(Y) \to \pi_k(X)$ are isomorphisms $(\forall k) \Leftrightarrow \pi_j(f) = 0\ \forall j \geq 1$ and $\pi_0(Y) \xleftrightarrow{1:1} \pi_0(X)$.

Killing homotopy classes via pushout:

Lemma 1.2. *$f\colon Y \to X$ k-connected map ($\pi_j(f) = 0$, $1 \leq j \leq k, \pi_0(Y) \xleftrightarrow{1:1} \pi_0(X)$) and let $w \in \pi_{k+1}(f)$*

$$w = \left[\begin{array}{ccc} S^{k} & \xrightarrow{q} & Y \\ {\scriptstyle j}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ D^{k+1} & \xrightarrow{Q} & X \end{array}\right]$$

Let Y' be the pushout of q and j, and let $f'\colon Y' \to X$ be the induced map. Then f' is k-connected and the induced map $\pi_{k+1}(f) \to \pi_{k+1}(f')$ is a surjective group homomorphism containing w in its kernel.

Sketch of proof. The map

$$\pi_l(f) \to \pi_l(f')$$

$$\left[\begin{array}{ccc} S^{l-1} & \xrightarrow{\widehat{q}} & Y \\ {\scriptstyle j}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ D^{l} & \xrightarrow{\widehat{Q}} & X \end{array}\right] \mapsto \left[\begin{array}{ccc} S^{l-1} & \xrightarrow{i_Y \circ \widehat{q}} & Y' \\ {\scriptstyle j}\big\downarrow & & {\scriptstyle f'}\big\downarrow \\ D^{l} & \xrightarrow{\widehat{Q}} & X \end{array}\right]$$

is actually a group homomorphism which is surjective, at least for $l \leq k+1$ ($\Rightarrow f$ is k-connected). w is mapped to 0:

$$\begin{array}{ccc} S^{k} & \xrightarrow{q} & Y \\ {\scriptstyle j}\big\downarrow & & {\scriptstyle i_Y}\big\downarrow \\ D^{k+1} & \xrightarrow{i_{D^{k+1}}} & Y' \end{array}$$

The map $\iota_{D^{k+1}}$ is null-homotopic since D^{k+1} is contractible, hence projecting the homotopy down using f' gives a homotopy of the diagram with the "trivial" one (where the horizontal arrows are constant maps). $\square$

Proposition 1.3. *$f\colon Y \to X$ $(k-1)$-connected map between finite CW-complexes, $k \geq 1$.*

(1) *If X is connected, $k \geq 2$, and $\pi_1(Y) \to \pi_1(X)$ is isomorphism, then $\pi_k(f)$ is a finitely generated $\mathbb{Z}\pi_1(Y)$-module.*
(2) *f can be made k-connected by attaching finitely many cells.*

1.2. **Performing surgery.**

Theorem 1.4 (H. Whitney)**.** *Let M^m and N^n be smooth manifolds and $f\colon M \to N$ be a continuous map.*

i) *If $2m \leq n$, then every C^0-neighbourhood of f contains an immersion.*
ii) *If $2m < n$, then every C^0-neighbourhood of f contains an embedding.*

Lemma 1.5. *$k \in \{0, \dots, [\frac{n-1}{2}]\}$, $f\colon M \to X$ continous map, f k-connected, $w \in \pi_{k+1}(f)$ represented by a commutative diagram*
$$\begin{tikzcd} S^k \arrow[r, "q"] \arrow[d] & M \arrow[d, "f"] \\ D^{k+1} \arrow[r, "Q"] & X \end{tikzcd}$$
with continuous maps q, Q. Assume q can be extended to a smooth embedding $S^k \times D^{n-k} \xrightarrow{\overline{q}} M$. Let M' be obtained from M by attaching a k-handle *along $\overline{q}$. Then f induces a k-connected map $M' \xrightarrow{f'} X$ bordant to f such that the induced map $\pi_{k+1}(f) \xrightarrow{\varphi} \pi_{k+1}(f')$ is surjective and satisfies $\varphi(w) = 0$.*

If $2k < n$: Whitney's theorem implies that there exists an embedding $q_0\colon S^k \to M$, near q. If $q_0\colon S^k \to M$ is an embedding and f is the base map of a normal map then q_0 can be extended as in the previous lemma.

1.3. **Regular homotopy classes of immersions.**

Theorem 1.6 (Hirsch-Smale)**.** *M closed m-dimensional manifold, N n-dimensional manifold.*

(1) *If $1 \leq m < n$, then the differential yields*
$$T\colon \pi_0(\mathrm{Imm}(M, N)) \overset{1:1}{\longleftrightarrow} \pi_0(\mathrm{Mono}(TM, TN)).$$
(2) *If $1 \leq m \leq n$ and M has a handlebody decomposition consisting only of k-handles, $k \leq n-2$, then*
$$T\colon \pi_0(\mathrm{Imm}(M, N)) \overset{1:1}{\longleftrightarrow} \mathrm{colim}_{a\to\infty}\, \pi_0(\mathrm{Mono}(TM \oplus \underline{\mathbb{R}}^a, TN \oplus \underline{\mathbb{R}}^a)).$$

Exercise 1.7.

(1) $\pi_0(\mathrm{Imm}(S^2, \mathbb{R}^3)) = 0$. It implies - among others - that a so-called *sphere eversion* exists: one can turn S^2 "inside out" using a regular homotopy.
(2) $\pi_0(\mathrm{Imm}(S^n, \mathbb{R}^{2n})) = \begin{cases} \mathbb{Z}_2 & \text{if } n \geq 3 \text{ is odd,} \\ \mathbb{Z} & \text{if } n \text{ is even or } n = 1. \end{cases}$

Theorem 1.8 (3.59)**.** *Consider*

- *A normal map $(\overline{f}, f)\colon TM \oplus \underline{\mathbb{R}}^a \to \xi$ covering $f\colon M \to X$,*
- *An element $w \in \pi_{k+1}(f)$ for $k \leq n-2$ ($n = \dim(M)$),*
- *$j\colon S^k \times D^{n-k} \to D^{k+1} \times D^{n-k}$,*
- *$Tj \oplus \vec{n}\colon T(S^k \times D^{n-k}) \oplus \underline{\mathbb{R}} \to T(D^{k+1} \times D^{n-k})$ covering j.*

Then
1) We can find a commutative diagram of vector bundles

$$\begin{tikzcd} T(S^k \times D^{n-k}) \oplus \underline{\mathbb{R}}^{a+b} \arrow[r, "\overline{q}"] \arrow[d] & TM \oplus \underline{\mathbb{R}}^{a+b} \arrow[d] \\ T(D^{k+1} \times D^{n-k}) \oplus \underline{\mathbb{R}}^{a+b-1} \arrow[r, "\overline{Q}"] & \xi \oplus \underline{\mathbb{R}}^b \end{tikzcd}$$

covering a commutative diagramm

$$\begin{tikzcd} S^k \times D^{n-k} \arrow[r, "q"] \arrow[d, "j"'] & M \arrow[d, "f"'] \\ D^{k+1} \times D^{n-k} \arrow[r, "Q"] & X \end{tikzcd}$$

such that the restriction to $D^{k+1} \times \{0\}$ represents w and q is an immersion.
2) The regular homotopy class of q is uniquely determined by the properties above and depends only on w and $(\overline{f}, f)$
3) If the regular homotopy class of q contains an embedding, then we can arrange q in assertion 1) to be an embedding. If $2k < n$, one can always find such an embedding.
4) Suppose that q is an embbeding.

- $W := M \times [0,1] \bigcup_q D^{k+1} \times D^{n-k}$ *where* $q \colon S^k \times D^{n-k} \to M = M \times \{1\}$
- $F \colon W \to X$ *the map induced by* $M \times [0,1] \xrightarrow{pr} M \xrightarrow{f} X$ *and* $Q \colon D^{k+1} \times D^{n-k} \to X$.

Then

(1) *After possibly stabilizing $\overline{f}$ the bundle maps $\overline{f}$ and $\overline{Q}$ induce a bundle map*

$$\overline{F} \colon TW \oplus \underline{\mathbb{R}}^{a+b-1} \to \xi \oplus \underline{\mathbb{R}}^b$$

covering $F \colon W \to X$.

(2) *We get a normal map $(\overline{F}, F)$ which extends $(\overline{f}, f) \colon TM \oplus \underline{\mathbb{R}}^{a+b} \to \xi \oplus \underline{\mathbb{R}}^b$*
(3) *$(\overline{f}', f') \colon TM' \oplus \underline{\mathbb{R}}^{a+b} \to \xi \oplus \underline{\mathbb{R}}^b$ obtained by restricting $(\overline{F}, F)$ to $M' := \partial W - M \times \{0\}$ is a normal map of degree 1.*
(4) *$(\overline{f}', f')$ is normally bordant to $(\overline{f}, f)$ with underlying manifold $M' = M - int(q(S^k \times D^{n-k})) \cup_q D^{k+1} \times S^{n-k-1}$.*

Proof. We have an extended commutative diagram

$$\begin{tikzcd} S^k \times D^{n-k} \arrow[r] \arrow[d, "j"'] \arrow[rr, bend left, "q"] & S^k \arrow[r, "q'"] \arrow[d, "j'"] & M \arrow[d, "f"] \\ D^{k+1} \times D^{n-k} \arrow[r] \arrow[rr, bend right, "Q"'] & D^{k+1} \arrow[r, "Q'"'] & X \end{tikzcd}$$

Since $D^{k+1} \times D^{n-k}$ is contractible, and

$$Q^*\xi \cong \underline{\mathbb{R}}^{n+a} = \underline{\mathbb{R}}^{n+1} \oplus \underline{\mathbb{R}}^{a-1} \cong T(D^{k+1} \times D^{n-k}) \oplus \underline{\mathbb{R}}^{a-1}.$$

there exists a bundle map $\overline{Q} \colon T(D^{k+1} \times D^{n-k)} \oplus \underline{\mathbb{R}}^{a-1} \to \xi$ covering Q.
From the fiberwise isomorphisms in the diagram of vector bundles, there exists a

unique bundle map $\overline{q}$ covering q such that the following diagram commutes

$$\begin{tikzcd}
T(S^k \times D^{n-k}) \oplus \underline{\mathbb{R}}^a \arrow[r, "\overline{q}"] \arrow[d] & TM \oplus \underline{\mathbb{R}}^a \arrow[d] \\
T(D^{k+1} \times D^{n-k}) \oplus \underline{\mathbb{R}}^{a-1} \arrow[r, "\overline{Q}"] & \xi \oplus \underline{\mathbb{R}}^b
\end{tikzcd}$$

For $k \leq n-2$ there exists an immersion $q_0 \colon S^k \times D^{n-k} \to M$, such that $(Tq_0, q_0) \colon T(S^k \times D^{n-k)} \to TM$ and $(\overline{q}, q)$ define the same element in $\operatorname{colim}_{a\to\infty} \pi_0(\operatorname{Mono}(T(S^k \times D^{n-k}) \oplus \underline{\mathbb{R}}^a, TM \oplus \underline{\mathbb{R}}^a)) \cong \pi_0(\operatorname{Imm}(S^k \times D^{n-k}, M))$.

By a cofibration argument, since j is an inclusion, $q_0 \simeq q$ and $f \circ q = Q \circ j \Rightarrow \exists Q_0 \colon D^{k+p} \times D^{n-k} \to X$ such that $Q_0 \circ j = f \circ q_0$ and $Q_0 \simeq Q$.

$$\begin{tikzcd}
 & T(S^k \times D^{n-k} \oplus \underline{\mathbb{R}}^{a+b}) \arrow[rr] \arrow[d] \arrow[ddl] & & TM \oplus \underline{\mathbb{R}}^{a+b} \arrow[d] \arrow[ddl] \\
 & S^k \times D^{n-k} \arrow[rr, "q"] \arrow[ddl, "j"] & & M \arrow[ddl, "f"] \\
T(D^{k+1} \times D^{n-k)} \oplus \underline{\mathbb{R}}^{a+b-1} \arrow[rr, "\overline{Q}"] \arrow[d] & & \xi \oplus \underline{\mathbb{R}}^b \arrow[d] & \\
D^{k+1} \times D^{n-k} \arrow[rr, "Q"] & & X &
\end{tikzcd}$$

To prove 4), note that the map given by

$$\begin{aligned} F \colon W &\to X \\ (x,t) \in M \times [0,1] &\mapsto f(x) \\ x \in D^{k+1} \times D^{n-k} &\mapsto Q(x) \end{aligned}$$

is well–defined. As before, possibly after stabilizing, we can construct

$$\overline{F} \colon TW \oplus \underline{\mathbb{R}}^{a+b-1} \to \xi \oplus \underline{\mathbb{R}}^b$$

such that the map $(\overline{F}, F)$ is normal and extends $(\overline{f}, f)$. By restricting $(\overline{F}, F)$ to M' we obtain the map $(\overline{f}', f') \colon TM' \oplus \underline{\mathbb{R}}^{a+b} \to \xi \oplus \underline{\mathbb{R}}^b$, which is normal of degree 1 and, by construction, it is normally bordant to $(\overline{f}, f)$. $\square$

Definition 1.9. Consider a normal map $(\overline{f}, f) : M \to N$ and an element $w \in \pi_{k+1}(f)$ for $k \leq n-2$ for $n = \dim(M)$. We call the normal map $(\overline{f}', f') \colon TM' \oplus \underline{\mathbb{R}}^{a+b} \to \xi \oplus \underline{\mathbb{R}}^b$ appearing in the last theorem, *the result of surgery on* $(\overline{f}, f)$ *and* w if it exists. Sometimes we call the step from $(\overline{f}, f)$ to $(\overline{f}', f')$ a *surgery step.*

Theorem 1.10 (3.61 - Making a normal map highly connected)**.** *Let X be a connected finite n-dimensional Poincaré complex and $\overline{f} \colon TM \oplus \underline{\mathbb{R}}^a \to \xi$ a normal map of degree 1 covering $f \colon M \to X$. Then we can carry out a finite sequence of surgery steps to obtain a normal map of degree 1*

$$\overline{g} \colon TN \oplus \underline{\mathbb{R}}^{a+b} \to \xi \oplus \underline{\mathbb{R}}^b$$

covering $g : N \to X$, such that $(\overline{f}, f)$ and $(\overline{g}, g)$ are normally bordant and g is k-connected where $n = 2k$ or $n = 2k+1$.

Surgery problem: *Suppose we have some normal map* $(\overline{f}, f)\colon M \to X$ *where* M *is a closed manifold and* X *is a finite Poincaré complex. Can we change* M *and* f *leaving* X *fixed by finitely many sugery steps to get a normal map* $(\overline{g}, g)$ *from a closed manifold* N *to* X *such that* g *is a homotopy equivalence?*