1. Manifold Atlas : Regensburg Surgery Blockseminar 2012
Surgery kernels (Filipp Levikov)

$f\colon M \to X$, X Poincaré complex, M manifold, degree one normal map

Definition 1.1.

$$K_k(M) = K_k(\widetilde{M}) = H_{k+1}(\widetilde{f}) = H_{k+1}(\mathrm{cyl}(\widetilde{f}), \widetilde{M})$$
$$K^k(M) = K^k(\widetilde{M}) = H^{k+1}(\widetilde{f}) = H^{k+1}(\mathrm{cyl}(\widetilde{f}), \widetilde{M})$$

Lemma 1.2.

(1) *f k-connected $\Leftrightarrow$ $K_i(M) = 0, i < k$ and f isomorphism on π_1*
(2) *f k-connected $\Rightarrow$ $K_k(M) \cong \pi_{k+1}(f) = \pi_{k+1}(\widetilde{f})$*
(3) *f is homotopy equivalence $\Leftrightarrow$ f isomorphism on π_1 and $K_*(M) = 0$*

Proof. Apply Hurewicz and Whitehead theorems □

Lemma 1.3.

(1) *$H_k(\widetilde{M}) = K_k(M) \oplus H_k(\widetilde{X})$ and the same for cohomology*
(2) *$\cap[M]\colon K^{n-k}(M) \xrightarrow{\cong} K_k(M)$*

Proof. Use the natural splittings which are given by the Umkehr maps

$$f_! : H^*(\widetilde{M}) \to H^*(\widetilde{X}) \text{ and } f^! : H_*(\widetilde{X}) \to H_*(\widetilde{M})$$

$$\begin{array}{ccccccc}
\longrightarrow K_k(M) & \longrightarrow & H_k(\widetilde{M}) & \underset{f^!}{\overset{\longrightarrow}{\longleftarrow}} & H_k(\widetilde{X}) & \xrightarrow{0} & 0 \text{ is} \\
\uparrow \cong & & \uparrow \cong & & \downarrow \cong & & \\
K^{n-k}(M) & \underset{\longrightarrow}{\longleftarrow} & H^{n-k}(\widetilde{M}) & \xleftarrow[f^*]{} & H^{n-k}(\widetilde{X}) & &
\end{array}$$

□

Lemma 1.4.

(1) *f k-connected, $K_k(M)$ is finitely generated projective $\mathbb{Z}\pi$-module*
(2) *$M \to X$, f k-connected, $K_k(M)$ is stably free and $K^k(M) \cong K_k(M)^*$.*

Corollary 1.5. *If $n = 2k$ or $n = 2k+1$, $K_i(M) = 0, i \leq k$, f is isomorphism on $\pi_1 \Rightarrow f$ is a homotopy equivalence.*

Lemma 1.6. *There is a natural $\mathbb{Z}\pi$-module homomorphism $t_k : \pi_{k+1}(f) \to I_k(M)$*

Idea.

(1) An element $\alpha \in \pi_{k+1}(f)$ can be represented by

$$\begin{array}{ccc}
S^k \times D^k & \xrightarrow{g} & M \\
\downarrow & & \downarrow \\
D^{k+1} \times D^k & \longrightarrow & X
\end{array}$$

and the normal data determines a unique regular homotopy class (cf. proof of the Surgery Step in talk 7) with a representative $\overline{g} \simeq_{\text{reg. htpy}} g$ and $\overline{g}\colon S^k \times D^k \looparrowright M$ a framed k-immersion.

(2) Alternatively define $I_{k+1}(f)$ to fit into the long exact sequence

$$\cdots I_{k+1}(f) \to I_k(M) \to \pi_k(M) \to \cdots$$

and identify $I_{k+1}(f) \cong \pi_{k+1}(f) \oplus \pi_{k+1}(\mathrm{BO}, \mathrm{BO}(n-k))$. The normal data implies that the framing obstruction of g lives in the latter summand (i.e. the normal bundle of g is stably trivial). The map t_k maps α first to the unique representative with vanishing framing obstruction. At the end compose with the boundary $I_{k+1}(f) \xrightarrow{\partial} I_k(M)$. (Cf. A. Ranicki, Geometric and Algebraic Surgery, Prop. 11.36)

Observation 1. : f k-connected, the two pairings on the surgery kernel coincide:

$$\begin{array}{ccc} K_k(M)\times K_k(M) & \xrightarrow{\lambda^{\text{alg}}} & \mathbb{Z}\pi \\ \downarrow & \nearrow_{\lambda^{\text{geom}}} & \\ I_k(M)\times I_k(M) & & \end{array}$$

So far we have the following data:

$$(K_k(M), \lambda, \mu)$$

In the following we are going to introduce the right algebraic language to describe this data.

Definition 1.7. R is *a* ring with involution, P finitely generated projective module over R . $\lambda: P\times P \to R$ seqsuilinear $\Leftrightarrow$

- $\lambda(p, r_1q_1 + r_2q_2) = r_1\lambda(p, q_1) + r_2\lambda(p, q_2)$
- $\lambda(r_1p_1 + r_2p_2, q) = \lambda(p_1, q)\overline{r}_1 + \lambda(p_2, q)r_2$

$S(P)$ denotes the additive group of sesquilinear pairingss on P and can be identified with

$$S(P) = \text{Hom}(P, P^*)$$
$$\lambda \mapsto (\psi_\lambda : x \mapsto \lambda(x, -))$$

In the following put $\varepsilon = \pm 1$. We have the ε-transposition involution operator.

$$T_\varepsilon : S(P) \longrightarrow S(P),$$
$$\lambda \mapsto T_\varepsilon$$

with $T_\varepsilon\lambda(x, y) = \varepsilon\overline{\lambda(y, x)}$.
ε-symmmetic group: $Q^\varepsilon(P) = \ker(1 - T_\varepsilon)$
ε-quadratic group: $Q_\varepsilon(P) = \text{coker}(1 - T_\varepsilon)$
The two are connected by the ε-symmetrisation map $Q_\varepsilon(P) \xrightarrow{1+T_\varepsilon} Q^\varepsilon(P)$

Definition 1.8. An ε-symmetric form over R is a pair (P, φ), P f.g projective R-module, $\varphi \in Q^\varepsilon(P)$. (P, φ) is called non-degenerate if $\varphi : P \xrightarrow{\cong} P^*$
Example 1.9.

(1) L f.g. projective module, $P = L \oplus L^*$,

$$\varphi : L\oplus L^* \xrightarrow{\begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}} (L\oplus L^*)^*$$

$\lambda((P, f), (P', f')) = f(P') + \varepsilon\overline{f'(P)}$
Notation: $H^\varepsilon(L)$, standard hyperbolic ε-symmetric form.

(2) $f : M^{2k} \to X^{2k}$ degree one normal data, f k-connected. $(K_k(M), \lambda^{\text{alg/geom}})$ is an ε-symmetric form over $\mathbb{Z}\pi$

(3) $(H_k(S^k \times S^k), \lambda) = H^{(-1)^k}(\mathbb{Z})$

Definition 1.10. An ε-quadratic form (P, ψ), P f.g. projective $[\psi] \in Q_\varepsilon(P)$. (P, ψ) is non-degenerated if $(1 + T_\varepsilon)\psi$ is non-degenerate (ε-symmetric form)

Lemma 1.11. *(P, ψ) can equivalently be described on (P, λ, μ), (P, λ) ε-symmetric form. $\mu : P \to Q_\varepsilon(R)$*

(1) $\mu(p+q) - \mu(p) - \mu(q) = [\lambda(p,q)] \in Q_\varepsilon(R)$
(2) $\mu(p) + \varepsilon\overline{\mu(p)} = \lambda(p,p)$
(3) $\mu(rp) = r\mu(p)\overline{r}$

Proof. In one direction the correspondence is given by $\lambda(p,q) = \psi(p)(q) + \varepsilon\overline{\psi(q)(p)}$, $\mu(p) = \psi(p)(p)$. See Ranicki, Algebraic and Geometric Surgery, Prop. 11.9 for details. □

Example 1.12.

(1) $f\colon M^{2k} \to X^{2k}$ degree 1 normal data, f k-connected. $(K_k(M), \lambda, \mu^{\mathrm{Wall}})$ is an ε-quadratic form.
(2) The (standard) non-singular hyperbolic ε-quadratic form is given by
$H_\varepsilon(L) = L \oplus L^*$, $\psi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
Clearly $(1+T_\varepsilon)H^\varepsilon(L) = H_\varepsilon(L)$.

Theorem 1.13. *$(f,\overline{f})\colon M^{2k} \to X^{2k}$, degree 1, $\overline{f}\colon TM \oplus \mathbb{R}^a \to \xi$ covering f, $k \geq 3$, f k-connected. Suppose there exist $u, v \in \mathbb{N}$ such that*

$$K_k(M) \oplus H_\varepsilon(\mathbb{Z}\pi^u) \cong H_\varepsilon(\mathbb{Z}\pi^v)$$

Then we can do a finite number of surgeries with effect $(\overline{f}', f')$ and f' a homotopy equivalence.

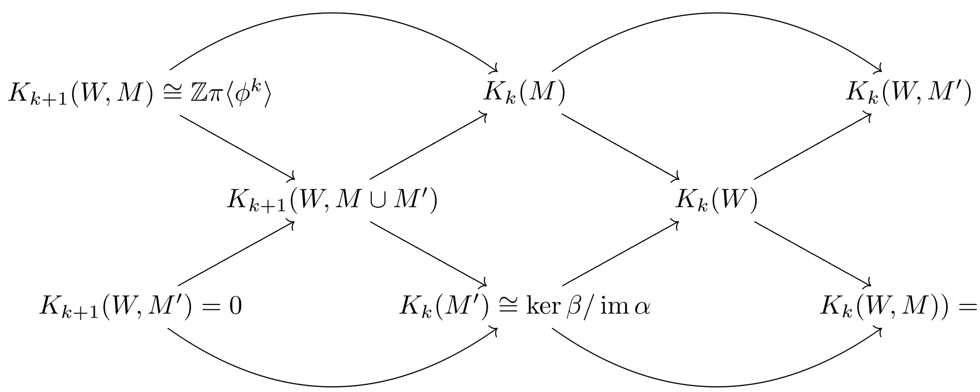
Proof. One can always achieve "stabilisation" geometrically (exercise!) by doing surgery on $0 \in \pi_k(\widetilde{f})$ to replace M by M'' with

$$K_k(M'') = K_k(M) \oplus H_\varepsilon(\mathbb{Z}\pi).$$

So w.l.o.g. assume $K_k(M) = H_\varepsilon(\mathbb{Z}\pi^v)$. There exists a (symplectic) basis $\{b_1, b_2, \dots b_n, c_1, \dots, b_n\}$ such that the quadratic form ψ on $K_k(M)$ is given by

$$\begin{pmatrix} 0 & \mathrm{id} \\ 0 & 0 \end{pmatrix}$$

In particular $\mu(b_n) = 0$ which by the Wall Embedding Theorem (talk 8) implies that b_n can be modified into an framed embedding. Hence we can do surgery to "kill" the class b_n. The diagramm shows that the effect of the surgery is "closer" to a homotopy equivalence since the kernel of the effect $K_k(M')$ is generated by $\{b_1, \dots, b_{n-1}, c_1, \dots, c_{n-1}\}$. Now repeat the procedure until the kernel is zero and apply Lemma 1.5.



M' is the effect of surgery on M, W is the trace and α and β are the maps in the top row. Here α is given by sending the generator to b_n and β by sending x to $\lambda(x, b_n)$ (exercise!). □