A guide to the calculation of the surgery obstruction groups for finite groups

Ian Hambleton and Laurence R. Taylor

We describe the main steps in the calculation of surgery obstruction groups for finite groups. Some new results are given and extensive tables are included in the appendix.

The surgery exact sequence of C. T. C. Wall **[68]** describes a method for classifying manifolds of dimension ≥ 5 within a given (simple) homotopy type, in terms of normal bundle information and a 4–periodic sequence of obstruction groups, depending only on the fundamental group and the orientation character. These obstruction groups $L_n^s(\mathbf{Z}G, w)$ are defined by considering stable isomorphism classes of quadratic forms on finitely generated free modules over $\mathbf{Z}G$ (n even), together with their unitary automorphisms (n odd).

Carrying out the surgery program in any particular case requires a calculation of the surgery obstruction groups, the normal invariants, and the maps in the surgery exact sequence. For fundamental group $G = 1$, the surgery groups were calculated by Kervaire–Milnor as part of their study of homotopy spheres:

$$L_n^s(\mathbf{Z}) = \ 8\mathbf{Z},\ 0,\ \mathbf{Z}/2,\ 0 \qquad \text{for } n = 0, 1, 2, 3 \pmod 4\ ,$$

where the non–zero groups are detected by the signature or Arf invariant, and the notation $8\mathbf{Z}$ means that the signature can take on any value $\equiv 0 \pmod 8$. The Hirzebruch signature theorem can be used to understand the signature invariant, and a complete analysis of the normal data was carried out by Milgram **[45]**, Madsen–Milgram **[43]** and Morgan–Sullivan **[48]**.

The theory of non–simply connected surgery has been used to investigate three important problems in topology:

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(i) *the spherical space form problem, or the classification of free finite group actions on spheres*

(ii) *the Borel and Novikov conjectures, or the study of closed aspherical manifolds and assembly maps*

(iii) *transformation groups, or the study of Lie group actions on manifolds.*

In the first problem, surgery is applied to manifolds with finite fundamental group and the surgery obstruction groups can be investigated by methods closely related to number theory and the representation theory of finite groups. In the second problem, the fundamental groups are infinite and torsion–free, and the methods available for studying the surgery obstruction groups are largely geometrical. The case of the n–torus was particularly important for its applications to the theory of topological manifolds. The third problem includes both finite group actions and actions by connected Lie groups. The presence of fixed point sets introduces many interesting new features.

In this paper we consider only $L_*(\mathbf{Z}G)$ for finite groups G. The Novikov conjectures and other topics connected with infinite fundamental groups are outside the scope of this article.

Before giving some notation, definitions and a detailed statement of results, it may be useful to list some general properties of the surgery obstruction groups for finite groups.

(1) *The groups $L_*(\mathbf{Z}G)$ are finitely generated abelian groups, the odd–dimensional groups $L_{2k+1}(\mathbf{Z}G)$ are finite, and in every dimension the torsion subgroup of $L_*(\mathbf{Z}G)$ is 2–primary.*

There is a generalization of the ordinary simply–connected signature, called the multi–signature [**68**, 13A], [**40**].

(2) *The multi–signature is a homomorphism $\sigma_G\colon L_{2k}(\mathbf{Z}G) \to R_{\mathbf{C}}^{(-)^k}(G)$ where $R_{\mathbf{C}}(G)$ denotes the ring of complex characters of G. The multi–signature has finite 2–groups for its kernel and cokernel.*

Complex conjugation acts as an involution on $R_{\mathbf{C}}(G)$, decomposing it as a sum of $\mathbf{Z}$'s from the real–valued (type I) characters, and a sum of free $\mathbf{Z}[\mathbf{Z}/2]$ modules generated by irreducible type II characters $\chi \neq \bar{\chi}$. The $(-1)^k$–eigenspaces of the complex conjugation action are denoted $R_{\mathbf{C}}^{(-)^k}(G)$.

The theory of Dress induction [**24, 25**] greatly simplifies the calculation of L–groups. A group G is called *p–hyperelementary* if $G = C \rtimes P$ where P is a p–Sylow subgroup and C is a cyclic group of order prime to p. Then G is determined by C, P and the structure homomorphism $t\colon P \to$

$Aut(C)$. Further, G is *p–elementary* if it is p–hyperelementary and t is trivial (equivalently $G = C \times P$).

(3) *$L_*(\mathbf{Z}G)$ can be calculated from knowledge of the L–groups of hyperelementary subgroups of G, together with the maps induced by subgroup inclusions.*

Moreover, one can calculate $L_*(\mathbf{Z}G) \otimes \mathbf{Z}_{(2)}$, $R_{\mathbf{C}}(G) \otimes \mathbf{Z}_{(2)}$ and $\sigma_G \otimes 1$ from the 2–hyperelementary subgroups and the maps between them. Since (1) and (2) imply that

$$\begin{array}{ccc} L_*(\mathbf{Z}G) & \xrightarrow{\sigma_G} & R_{\mathbf{C}}(G) \\ \downarrow & & \downarrow \\ L_*(\mathbf{Z}G) \otimes \mathbf{Z}_{(2)} & \xrightarrow{\sigma_G \otimes 1} & R_{\mathbf{C}}(G) \otimes \mathbf{Z}_{(2)} \end{array}$$

is a pull–back, Dress's work computes $L_*(\mathbf{Z}G)$ in terms of representation theory and the L–theory of 2–hyperelementary groups. For this reason, most of the calculational work has been devoted to the 2–hyperelementary case.

These general properties are fine until one needs more precise information for computing surgery obstructions. An early result of Bak and Wall (worked out as an example in Theorem 10.1) is that for G of *odd order*

$$L^s_n(\mathbf{Z}G) = \ \Sigma \oplus 8\mathbf{Z},\ 0,\ \Sigma \oplus \mathbf{Z}/2,\ 0 \qquad \text{for } n = 0, 1, 2, 3 \pmod 4\ .$$

The terms $\Sigma = \oplus 4(\chi \pm \bar{\chi})$ comes from the multisignatures at type II characters, and the term $\mathbf{Z}$ is the summand of $R_{\mathbf{C}}(G)$ generated by the trivial character. The term $\mathbf{Z}/2$ is detected by the ordinary Arf invariant.

Another nice case is $G = C \times P$, where C is a cyclic 2–group and P has odd order (this includes arbitrary cyclic groups as well as the p–hyperelementary groups G for p odd). Assuming C is non–trivial, we have:

$$L^s_n(\mathbf{Z}G) = \ \Sigma \oplus 8\mathbf{Z} \oplus 8\mathbf{Z},\ 0,\ \Sigma \oplus \mathbf{Z}/2,\ \mathbf{Z}/2 \qquad \text{for } n = 0, 1, 2, 3 \pmod 4\ .$$

The signature group again has two sources, the term $\Sigma = \oplus 4(\chi \pm \bar{\chi})$ from the type II characters and the two $\mathbf{Z}$'s coming from the type I characters (just the trivial character and the linear character which sends a generator to -1). The $\mathbf{Z}/2$ in dimension 2 is the ordinary Arf invariant and the $\mathbf{Z}/2$ in dimension 3 is a "codimension one" Arf invariant. The special case $G = \mathbf{Z}/2^r$ is worked out in Example 11.1.

Many geometric results have been obtained just from the vanishing of the odd–dimensional L–groups of odd order groups, but unfortunately

$L_n^s(\mathbf{Z}G)$ is usually not zero, and the torsion subgroup can be complicated (for example, even when G is a group of odd order times an abelian 2–group).

Nor does it help to relax the Whitehead torsion requirements, and allow surgery just up to homotopy equivalence. For example, the group $L^h_{2k}(\mathbf{Z}[\mathbf{Z}/2^r])$ has torsion subgroup $\big([2(2^{r-2}+2)/3]-[r/2]-\epsilon\big)\mathbf{Z}/2$, where $\epsilon = 1$ if k is even and 0 if k is odd **[12**, Thm.A]. The notation $[x]$ means the greatest integer in x. The source of this torsion is $D(\mathbf{Z}G) \subseteq \tilde{K}_0(\mathbf{Z}G)$, a part of the projective class group that is often amenable to calculation **[50]**.

The torsion subgroup of $L_n(\mathbf{Z}G)$ can also involve the ideal class groups of the algebraic number fields in the centre of the rational group algebra $\mathbf{Q}G$, and the computation of ideal class groups is a well–known and difficult problem in number theory. Another major complication is that computing the surgery obstruction groups often requires information about the Whitehead groups $Wh(\mathbf{Z}G)$, the algebraic home for the theory of Whitehead torsion.

Here the problem is that the torsion subgroup $SK_1(\mathbf{Z}G)$ of $Wh(\mathbf{Z}G)$ is highly non–trivial **[49]**. In particular, both the first optimistic claims for the Whitehead groups of abelian groups (tentatively quoted by Milnor in **[47]**) and Wall's conjecture **[75**, p.64,5.1.3] about the Tate cohomology of Whitehead groups, turned out to be incorrect.

In spite of these complications, the L–groups can be effectively computed in many cases of interest. The approach presented here (following the procedure established by Wall in **[67]**–**[75]**) will be to try and reduce the compution of $L_*(\mathbf{Z}G)$ to specific and independent questions in number theory and representation theory. From the statement of results in Section 2, we hope that the reader can get an overview of present knowledge, and useful references for further investigation. In the rest of the paper, we describe the main steps in the calculation and work out some relatively easy examples.

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