Embeddings of k-complexes in $2k$-manifolds and minimum rank of partial symmetric matrices*

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Abstract

Let K be a k-dimensional simplicial complex having n faces of dimension k and M a closed $(k-1)$-connected PL $2k$-dimensional manifold. We prove that *for $k\ge 3$ odd K embeds into M if and only if there are*

• *a skew-symmetric $n\times n$-matrix A with $\mathbb{Z}$-entries whose rank over $\mathbb{Q}$ does not exceed* $\operatorname{rk} H_k(M;\mathbb{Z})$,

• *a general position PL map $f: K\to\mathbb{R}^{2k}$, and*

• *a collection of orientations on k-faces of K*

such that for any nonadjacent k-faces σ,τ of K the element $A_{\sigma,\tau}$ equals to the algebraic intersection of $f\sigma$ and $f\tau$.

We prove some analogues of this result including those for $\mathbb{Z}_2$- and $\mathbb{Z}$-embeddability. Our results generalize the Bikeev-Fulek-Kynčl-Schaefer-Stefankovič criteria for the $\mathbb{Z}_2$- and $\mathbb{Z}$-embeddability of graphs to surfaces, and are related to the Harris-Krushkal-Johnson-Paták-Tancer criteria for the embeddability of k-complexes into $2k$-manifolds.

Contents

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1 Introduction and main results

1.1 Introduction

The study of graph drawings on 2-dimensional surfaces is an active area of mathematical research. Higher-dimensional generalization is classical and has attracted much attention recently. A major problem asks if there exists an algorithm for recognizing the embeddability of k-dimensional complexes into a *given* $2k$-dimensional manifold. A quick algorithm based on a beautiful mathematical result is (as always) preferable. If $k = 1$, or if the manifold is a sphere (or a ball or $\mathbb{R}^{2k}$), then there is a classical algorithm, see e.g. survey [MTW, §1]. For $k > 1$ and a closed manifold different from S^{2k} no algorithm is known.

We shorten 'k-dimensional (face, manifold, etc)' to 'k- (face, manifold, etc)'.

A general position PL map $f : K \to M$ of a graph to a 2-surface (a.k.a. a graph drawing on M) is called a $\mathbb{Z}_2$**-embedding** (a.k.a. Hanani-Tutte drawing) if $|f\sigma \cap f\tau|$ is even for any pair σ, τ of non-adjacent (a.k.a. independent) edges.

Analogously, a general position PL map $f : K \to M$ of a k-complex to a PL $2k$-manifold is called a $\mathbb{Z}_2$**-embedding** if $|f\sigma \cap f\tau|$ is even for any pair σ, τ of nonadjacent k-faces.

$\mathbb{Z}$-embedding is defined analogously in §1.3.

Algorithms for recognizing the $\mathbb{Z}_2$- and $\mathbb{Z}$-embeddability of k-complexes to given $2k$-manifold are also interesting.

Our main results are criteria for the $\mathbb{Z}_2$- and $\mathbb{Z}$-embeddability of k-dimensional complexes to $2k$-dimensional manifolds (Theorems 1.2.3 and 1.3.4). These criteria

• give embeddability criteria by Theorem 1.3.1.a;

• reduce embeddability to finding minimal rank of 'partial matrix' (and to related problems); this is extensively studied in computer science, see e.g. [Ko21] and surveys [MC, NKS];

• generalize criteria for the $\mathbb{Z}_2$- and $\mathbb{Z}$-embeddability of graphs to surfaces obtained in Bikeev's paper [Bi21] using ideas of Fulek-Kynčl-Schaefer-Stefankovič [SS13, FK19];

• are related to the Harris-Krushkal-Johnson-Paták-Tancer criteria for the embeddability of k-complexes into $2k$-manifolds (as explained in Remarks 1.1.1.b and 1.3.7).

We also obtain interesting Theorems 2.4.1, 2.4.2, and Corollaries 1.2.1, 1.2.2, 1.3.2, 1.3.3 and 1.3.5; some of them are easier-to-state and so are presented before the criteria. The cases $k = 1$ of all our results are known ([Bi21, Theorems 1.1 and 1.4], [SS13, Lemma 1]; Theorem 2.4.2 for $k = 1$ follows from [SS13, Lemma 3]).

Remark 1.1.1. (a) *Motivation.* The notions of $\mathbb{Z}_2$- and $\mathbb{Z}$-embedding naturally appeared in the studies of embeddings. Some proofs of the non-embeddability of k-complexes into $\mathbb{R}^{2k}$ actually show that these complexes are not $\mathbb{Z}_2$- or $\mathbb{Z}$-embeddable to $\mathbb{R}^{2k}$. See e.g. surveys [Sk18, Lemma 1.4.3], [Sk14, Theorem 1.4] and [KS21, Remark 1.2.d]. Some constructions of embeddings have constructions of $\mathbb{Z}$-embeddings as a natural intermediate step allowing to structure the proof. See e.g. comments after Theorem 1.3.1, [FKT, §2] and survey [Sk06, §4]. The property of being a $\mathbb{Z}_2$- or $\mathbb{Z}$-embedding is *stable*, i.e. is preserved under small enough perturbation of a map (as opposed to the property of being an embedding). The notion of $\mathbb{Z}_2$-embedding is most actively studied for the case of graphs drawings on surfaces, see survey [Sc13] and [SS13, FK19, Bi21]. Analogous remarks hold for *almost embeddings* (defined e.g. in [FKT, §4], [PT19, §1]).

(b) *Known higher-dimensional criteria.* Criteria for the embeddability of k-complexes in m-manifolds for $2m \geq 3k+3$ are given in [Ha69, Theorem 1 and Corollaries 6, 7, 8], in terms of isovariant maps or cohomology obstructions. Assume further that $m = 2k \geq 6$. Such criteria in a different equivalent form involving intersection cocycles are given in [Jo02]; see exposition in §2.3, in (EH) of Proposition 2.5.1, and in Remark 2.5.6.b; cf. [La70]. A criterion in terms of solvability of a system of quadratic Diophantine equations is the conjunction of [PT19, Theorems 1, 4 and 15] (cf. Theorem 1.3.4 and Remark 1.3.7.a). A new formula for the cohomology obstruction is given implicitly in [Kr00, Theorem 3.2] and explicitly in [PT19, Theorem 1][1], see exposition in Lemma 2.5.5. An algorithmic criterion for $\mathbb{Z}_2$-embeddability in terms of the cohomology obstruction is [PT19, Proposition 16, Theorem 10.i]. See footnote 3. For the Kühnel problem see [PT19, KS21] and the references therein.

Notation and conventions.

Denote by $|S|_2 \in \mathbb{Z}_2$ the number of elements modulo 2 in a finite set S.

Denote by Δ_N^k the union of k-faces of N-simplex.

A PL manifold M is called $(k-1)$-**connected** if for any $j = 0, 1, \ldots, k-1$ any continuous map $S^j = \Delta_{j+1}^j \to M$ extends to a continuous map $\Delta_{j+1}^{j+1} \to M$.

We work in the piecewise-linear (PL) category. Unless otherwise indicated, $k \geq 1$ is any integer, K a (simplicial) k-complex having n faces of dimension k, and M is any compact $(k-1)$-connected PL $2k$-manifold (possibly, with boundary).

In this paragraph R is $\mathbb{Z}_2$ or $\mathbb{Z}$. Let $H_k(X; R)$ be the k-dimensional **homology group** of a complex X with coefficients in R, defined to be the group of *homology classes* of k-*cycles* in X with coefficients in R. For a simple accessible to non-specialists definition see [IF, §2], [Sk20, §6, §10]. Let $H_k(M; R) := H_k(X; R)$ for any triangulation X of M.

We omit $\mathbb{Z}_2$-coefficients from the notation of (co)homology groups.

1.2 $\mathbb{Z}_2$-embeddings

Corollary 1.2.1. *(a) There is an algorithm checking the $\mathbb{Z}_2$-embeddability of k-complexes to M.*

(b) For any s there is a k-complex having no $\mathbb{Z}_2$-embedding (and so no embedding) to the connected sum of s copies of $S^k \times S^k$. As an example one can take the disjoint union of $s+1$ copies of Δ_{2k+2}^k.

Part (a) follows from Theorem 1.2.3 and Lemma 2.3.2 for $M = \mathbb{R}^{2k}$. Part (b) follows from Theorem 2.4.1.

Definition of the modulo 2 intersection form. For k-cycles α, β modulo 2 in (any triangulation of) M take close (or homotopic) 'general position' modulo 2 k-cycles α', β'. Let

$$\alpha \cap_M \beta := |\alpha' \cap \beta'|_2 \in \mathbb{Z}_2.$$

This is well-defined, i.e., is independent of α', β' for given α, β. This is also independent of the choice of α, β within their homology classes. So we denote this by $[\alpha] \cap_M [\beta] \in \mathbb{Z}_2$, where $[\alpha], [\beta] \in H_k(M)$, and denote by $\cap_M : H_k(M) \times H_k(M) \to \mathbb{Z}_2$ the modulo 2 intersection form of M. For details and references see [IF, §2]. (If M is closed, then $\operatorname{rk} \cap_M = \operatorname{rk} H_k(M)$ by Poincaré duality.)

[1] See Remark 2.5.2.b. Theorem 3.2 of [Kr00] concerns the case of embedding onto spines, $k = 2$, rational coefficients, and embeddings (not almost embeddings as in [PT19, Theorem 1]). Generalizations to arbitrary embeddings, to $k \geq 3$, and to integer coefficients, are straightforward. For $k \geq 3$ almost embeddability is equivalent to embeddability by Theorem 1.3.1.a (it would be interesting to see whether the obstructions of [Kr00] are obstructions to almost embeddability, not just to embeddability). The completeness of this obstruction follows by seeing that the obstruction of [Kr00, Theorem 3.2] equals to the obstruction of [Jo02]; the completeness was explicitly proved only as [PT19, Theorem 4]. Since the van Kampen obstruction has order 2, the rational van Kampen obstruction of [Kr00, Theorem 3.2] is zero.

For $R = \mathbb{Z}$ or $\mathbb{Z}_2$ a bilinear form $q : V \times V \to R$ on a $\mathbb{Z}_2$-vector space or $\mathbb{Z}$-module V is called **even** if $q(v, v)$ is even for any $v \in V$, and is **odd** otherwise. A symmetric matrix with $\mathbb{Z}_2$- or $\mathbb{Z}$-entries is **even** (for the case of $\mathbb{Z}_2$ a.k.a. alternate) if its diagonal contains only even entries, and is **odd** otherwise. The **type** of a bilinear form or a symmetric matrix is its being even or odd.

Corollary 1.2.2. *The $\mathbb{Z}_2$-embeddability of a given k-complex to M depends only on the rank and the type of $\cap_M$.*

This follows from Theorem 1.2.3 and Remark 1.2.4.a.

The complex K is called **compatible modulo 2** to a symmetric $n \times n$-matrix A with $\mathbb{Z}_2$-entries if there is a general position PL map $f : K \to \mathbb{R}^{2k}$ such that

(C_f) $\quad A_{\sigma,\tau} = |f\sigma \cap f\tau|_2$ for any non-adjacent k-faces (for $k = 1$ edges) σ, τ of K.

Compatibility modulo 2 is algorithmically decidable by Lemma 2.3.2 for $M = \mathbb{R}^{2k}$.

Theorem 1.2.3 (proved in §2.2, §2.3)**.** *There is a $\mathbb{Z}_2$-embedding $K \to M$ if and only if K is compatible modulo 2 to a matrix A such that* $\operatorname{rk} A \le \operatorname{rk} \cap_M$ *and A is even/odd if $\cap_M$ is even/odd.*

See more equivalent conditions in Proposition 2.5.1.

Remark 1.2.4. (a) Recall that

• there are two isomorphism classes of non-degenerate symmetric bilinear forms over $\mathbb{Z}_2$ of given rank: odd forms and (for even rank) even forms [IF, Theorem 6.1];

• $(k-1)$-connected smooth $2k$-manifolds with odd forms exist only for $k = 1, 2, 4$ (see a folklore proof in [KS21, footnote 1]; the PL analogue presumably holds).

(b) Is there a polynomial (in n) algorithm for checking the $\mathbb{Z}_2$-embeddability of k-complexes to a fixed M for $k > 1$? By (a) and Corollary 1.2.2 it suffices to answer this question for M being a connected sum of several copies of $S^k \times S^k$ (for $\cap_M$ even) or of $\mathbb{C}P^2$, of $\mathbb{H}P^2$ (for $\cap_M$ odd). For $k = 1$ see [Bi21, Remark 1.2.b].

1.3 $\mathbb{Z}$-embeddings and embeddings

In this section we assume that M is orientable (until Corollary 1.3.3 there is no need to assume that M is $(k-1)$-connected).

Let $f : K \to M$ be a general position PL map.

Then preimages $y_1, y_2 \in K$ of any double point $y \in M$ lie in the interiors of k-faces. By general position, f is 'linear' on some neighborhood U_j of y_j for each $j = 1, 2$. Given orientation on the edges, we can take a basis of 2 vectors formed by oriented fU_1, fU_2. The *intersection sign of y* is the sign ± 1 of the basis. The **algebraic intersection number** $f\sigma \cdot f\tau \in \mathbb{Z}$ for non-adjacent oriented k-faces σ, τ is defined as the sum of the intersection signs of all intersection points from $f\sigma \cap f\tau$.

An example of a $\mathbb{Z}_2$-embedding which is not a $\mathbb{Z}$-embedding is shown in Fig. 1, right.

The map f is called a **$\mathbb{Z}$-embedding** if $f\sigma \cdot f\tau = 0$ for any pair σ, τ of non-adjacent k-faces. (The sign of $f\sigma \cdot f\tau$ depends on an arbitrary choice of orientations for σ, τ and on the order of σ, τ, but the condition $f\sigma \cdot f\tau = 0$ does not.)

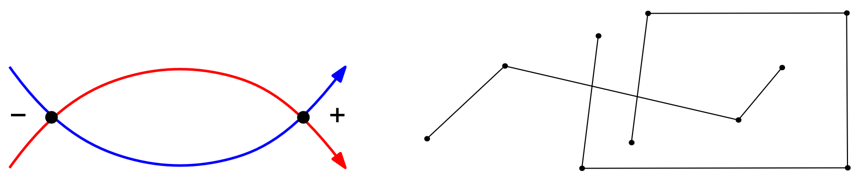

Figure 1: Two curves intersecting at an even number of points the sum of whose signs is zero (left) or non-zero (right).

Theorem 1.3.1. *(a) A complex of dimension $k \ge 3$ is PL embeddable into a simply connected PL $2k$-manifold M if and only if the complex is $\mathbb{Z}$-embeddable to M.*

(b) For each $k \ge 2$ there is a 2-complex $\mathbb{Z}_2$-embeddable but not $\mathbb{Z}$-embeddable to $\mathbb{R}^{2k}$.

(c) There is a 2-complex $\mathbb{Z}$-embeddable but not embeddable to $\mathbb{R}^4$.

Comments on the proof. Part (a) is a convenient intermediate step (both for this paper and for [PT19]) allowing to construct embeddings from $\mathbb{Z}$-embeddings, and to describe relation of the proof to known results and methods. Part (a) is essentially classical. For $M = \mathbb{R}^{2k}$ it is proved in [vK32, Sh57, Wu58]; for a simple exposition see [FKT, §2], [Sk06, §4]. The general case is proved in the same way, just note that in [FKT, Lemma 4, 5 and application of the Whitney trick in the proof of Theorem 3] $\mathbb{R}^{2k}$ could be replaced by M. See details in [Jo02, Corollary 2 and Theorem 4]. The proofs of [Jo02] mentioned here and in Remark 2.3.3 are written for 'smooth' maps of complexes but work for PL maps. The analogue of (a) for $k = 1$ is correct (because a compact simply connected 2-manifold is a sphere, and by the Hanani-Tutte Theorem, see e.g. [Sk18, Theorem 1.5.3]).

Part (b) is proved in [Me06, Example 3.6].

Part (c) is proved in [FKT] (although it is stated there in a different equivalent form), see a simpler proof in [AMS+, §2.2]. □

We conjecture that *the analogue of Corollary 1.2.1.a is correct for $k \ge 2$, M orientable and '$\mathbb{Z}_2$-embeddability' replaced by '$\mathbb{Z}$-embeddability' (or, equivalently for $k \ge 3$, by 'embeddability').*

For some K, M an integer analogue of Theorem 1.2.3 is obtained by replacing residues modulo 2 by integers. The complex K is called **compatible** to an $n \times n$-matrix A with $\mathbb{Z}$-entries if there is a general position PL map $f : K \to \mathbb{R}^{2k}$ such that for some collection of orientations on k-faces (for $k = 1$ edges) of K we have

$$(C_{f,\mathbb{Z}}) \qquad A_{\sigma,\tau} = f\sigma \cdot f\tau \quad \text{for any non-adjacent } k\text{-faces } \sigma, \tau \text{ of } K.$$

It is not clear whether compatibility is algorithmically decidable (in spite of Lemma 2.3.4 for $M = \mathbb{R}^{2k}$).

In this paper by the rank of a matrix with integer entries, or of a bilinear form on a $\mathbb{Z}$-module, we mean its rank over $\mathbb{Q}$.

Corollary 1.3.2. *(a) A 4-complex embeds into $\mathbb{H}P^2$ if and only if the complex is compatible to a rank 1 symmetric $n \times n$-matrix A with $\mathbb{Z}$-entries, whose any diagonal element is the square of an integer.*

(b) A 2-complex has a $\mathbb{Z}$-embedding to $\mathbb{C}P^2$ if and only if the complex is compatible to a matrix as in (a).

Corollary 1.3.2 follows by Theorem 1.3.4 and Lemma 2.1.7 (for (a) we also need Theorem 1.3.1.a). Corollaries 1.3.2.a, 1.3.3.a and Theorem 1.3.4 could perhaps be useful for [PT19, Problem 22].

Definition of the integer intersection form. For integer k-cycles α, β in (any triangulation of) M take close (or homotopic) 'general position' integer k-cycles α', β'. Let

$$\alpha \cap_{M,\mathbb{Z}} \beta := \alpha' \cdot \beta \in \mathbb{Z}.$$

This is well-defined, i.e., is independent of α', β' for given α, β. This is also independent of the choice of α, β within their homology classes. So we denote this by $[\alpha] \cap_{M,\mathbb{Z}} [\beta] \in \mathbb{Z}$, where $[\alpha], [\beta] \in H_k(M;\mathbb{Z})$, and denote by $\cap_{M,\mathbb{Z}} : H_k(M;\mathbb{Z}) \times H_k(M;\mathbb{Z}) \to \mathbb{Z}$ the integer intersection form of M. For details and references see [IF, §2]. Clearly, the product of a torsion element of $H_k(M;\mathbb{Z})$ with any element is zero. Hence $\cap_{M,\mathbb{Z}}$ can be regarded as a form $H_k(M;\mathbb{Z})/T \times H_k(M;\mathbb{Z})/T \to \mathbb{Z}$, where T is the torsion subgroup. By $\operatorname{rk} H_k(M;\mathbb{Z})$ and $\operatorname{rk} \cap_{M,\mathbb{Z}}$ we denote the rank over $\mathbb{Q}$ of $H_k(M;\mathbb{Z})/T$ and of $\cap_{M,\mathbb{Z}}$ defined on this quotient. (If M is closed, then $\operatorname{rk} \cap_{M,\mathbb{Z}} = \operatorname{rk} H_k(M;\mathbb{Z})/T$ by Poincaré duality. A standard fact is that $H_k(M;\mathbb{Z})$ is a free abelian group, i.e., $T = 0$, when M is $(k-1)$-connected.)

Corollary 1.3.3. *(a) For k odd and M closed there is a $\mathbb{Z}$-embedding $K \to M$ if and only if K is compatible to a skew-symmetric matrix A such that $\operatorname{rk} A \le \operatorname{rk} \cap_{M,\mathbb{Z}}$.*

(b) The $\mathbb{Z}$-embeddability of given k-complex to M depends only on $\cap_{M,\mathbb{Z}}$.

Corollary 1.3.3 follows by Theorem 1.3.4; for (a) we also need Lemma 2.1.6 and the fact that any unimodular skew-symmetric bilinear form over $\mathbb{Z}$ is isomorphic to the symplectic form [IF, Proposition 6.2].

For general k, M an integer analogue of Theorem 1.2.3 is more complicated.

Let V be an abelian group (or, equivalently, a $\mathbb{Z}$-module). The complex K is called **realizable by a bilinear form $I : V \times V \to \mathbb{Z}$ and a collection of elements $y_\sigma \in V$ indexed by k-faces σ of K** if there are a general position PL map $f : K \to \mathbb{R}^{2k}$ and orientations on k-faces of K such that

$$(R_{f,\mathbb{Z}}) \qquad I(y_\sigma, y_\tau) = f\sigma \cdot f\tau \quad \text{for any non-adjacent } k\text{-faces } \sigma, \tau \text{ of } K.$$

Comment. Here is the matrix form of the above definition. The complex K is called *realizable by an $s \times s$-matrix I of integers and a collection of vectors $y_\sigma \in \mathbb{Z}^s$ indexed by k-faces of K* if there are a general position PL map $f : K \to \mathbb{R}^{2k}$, and orientations on k-faces of K such that

$$(R_{f,\mathbb{Z}}) \qquad y_\sigma^T I y_\tau = f\sigma \cdot f\tau \quad \text{for any non-adjacent } k\text{-faces } \sigma, \tau \text{ of } K.$$

Since the system $(R_{f,\mathbb{Z}})$ of Diophantine equations with coefficients $I, f\sigma \cdot f\tau$ and variables $\{y_\sigma\}$ is quadratic, it is not clear whether its solvability is algorithmically decidable.

Theorem 1.3.4 (proved in §2.2, §2.3)**.** *There is a $\mathbb{Z}$-embedding $K \to M$ if and only if K is realizable by $\cap_{M,\mathbb{Z}}$ and some collection of homology classes $y_\sigma \in H_k(M;\mathbb{Z})$.*

Corollary 1.3.5. *Assume that k is even, M is closed, and $\cap_{M,\mathbb{Z}}$ is odd indefinite with positive and negative ranks r_+ and r_-. There is a $\mathbb{Z}$-embedding $K \to M$ if and only if K is realizable by a collection of vectors $y_\sigma \in \mathbb{Z}^{r_+ + r_-}$ and the square matrix of size $r_+ + r_-$ having r_+ units on the diagonal, r_- elements '-1' on the diagonal and zeros outside the diagonal.*

This follows by Theorem 1.3.4 and classification of symmetric unimodular odd indefinite bilinear forms over $\mathbb{Z}$ [IF, Theorem 6.3.a].

Conjecture 1.3.6. (a) For k odd there is a $\mathbb{Z}$-embedding $K \to M$ if and only if K is compatible to a skew-symmetric matrix A such that $\operatorname{rk} A \le r := \operatorname{rk} \cap_{M,\mathbb{Z}}$ and every $r \times r$ minor of A is divisible by the determinant of the matrix of $\cap_{M,\mathbb{Z}}$ in some basis of $H_k(M;\mathbb{Z})$.

(This would follow from (b) and Theorem 1.3.4.)

(b) Suppose that A, H are skew-symmetric matrices with $\mathbb{Z}$-entries, H is nondegenerate, $\operatorname{rk} A \le \operatorname{rk} H$, and every $\operatorname{rk} H \times \operatorname{rk} H$ minor of A is divisible[2] by $\det H$. Then there is a matrix Y with $\mathbb{Z}$-entries such that $A = Y^T HY$.

(c) Both conditions of Corollary 1.3.5 are equivalent to K being compatible to an odd indefinite symmetric matrix A whose positive and negative ranks do not exceed those of $\cap_{M,\mathbb{Z}}$.

Remark 1.3.7 (Relation to known results)**.** (a) For $k \ge 3$ and M closed

- Corollary 1.2.1.a is [PT19, Theorem 10.i];
- Corollary 1.3.3.b is essentially the same as [PT19, Proposition 8], cf. (e);
- Corollary 1.2.2 is easily deduced from [PT19, Proposition 16], and Theorem 1.3.4 is easily deduced from Theorem 1.3.1.a and the conjunction of [PT19, Theorems 1, 4 and 15].[3]

Our statements and proofs are simpler than those of [PT19]. This is mostly because we do not use Paták-Tancer version of the cohomology obstruction, but present a simpler equivalent statement of the necessary condition (compare the definition of realizability to the definition of ω in [PT19] exposed in §2.5). This simpler equivalent statement is an algebraically invariant (i.e., basis-free) version of the conjunction of [PT19, Theorems 1, 4 and 15]. Having it explicitly stated allows us obtain further results with simpler proofs.

(b) The notion of compatibility (modulo 2) appeared in [FK19, Bi21]. The definition of realizability (modulo 2) appeared implicitly in [SS13, §2.2, equality (3)], [FK19, §3.1, equality (1)], [PT19, equality (7) in §3], [Bi21, §2, Proof of the implication ($\Leftarrow$) of Theorem 1.4], and explicitly in our discussions with A. Bikeev. The collection of vectors y_σ from [SS13, FK19] and this paper is essentially the same as the homomorphism ψ from [PT19].

(c) Our constructions of $\mathbb{Z}_2$- and $\mathbb{Z}$-embeddings in Theorems 1.2.3 and 1.3.4 use known algebraic Lemma 2.1.1 and the known construction of a map inducing a given homomorphism in homology, cf. Lemma 2.5.3.[4] Our construction of matrix A and vectors y_σ in Theorems 1.2.3 and 1.3.4 use Lemmas 2.3.2 and 2.3.4 on the Harris-Johnson cohomology obstructions (see Remark 2.3.3), and the modification of a given map as in the statement of [PT19, Theorem 13] (exposed in Lemma 2.5.5).

(d) Geometric proofs (closer to [Bi21]) of Theorems 1.2.3, 1.3.4, and 2.4.2 could perhaps be obtained using a handle decomposition of M.

(e) It would be interesting to know if *for $k \ge 3$ any two $(k-1)$-connected PL $2k$-manifolds are PL homeomorphic if they have isomorphic intersection forms and boundaries PL homeomorphic to S^{2k-1}* (cf. classification of 'almost closed' manifolds [Wa62, p. 170]; pre-H-spaces and H-spaces are defined on p. 168 and 169). This statement implies Corollary 1.3.3.b for $k \ge 3$ and M closed (so it implies [PT19, Proposition 8]).

[2]The divisibility is automatic when $\operatorname{rk} A < \operatorname{rk} H$.

[3] The cited statements of [PT19] (as opposed to Corollary 1.2.2 and Theorem 1.3.4) involve cumbersome definition of a cohomology obstruction. We have to warn the reader that this definition is based on meaningless formulas $\xi(\sigma\times\tau) = (-1)^k\xi(\tau\times\sigma)$ and $\nu_f(\sigma\times\tau) = f(\sigma)\cdot f(\tau)$ [PT19, p. 2 and 7]. See an explanation why they are meaningless in [KS21, Remark 3.3, Critical remarks, (6)]; see a meaningful exposition in Remark 2.5.6.a referring to §2.3. The authors of [PT19] prepared an improved version in 2020 (according to critical remarks by A. Skopenkov). When that (or even more improved) version will be publicly available, critical remarks on [PT19] here will be replaced by remarks on the update (i.e., hopefully will be just deleted).

[4]For this reason a part of our constructions is similar to [PT19, §4, proof of Theorem 4, step 1] (whose authors presumably rediscovered this known construction, and so did not refer to it).

2 Proofs

2.1 Linear algebraic lemmas

Let V be a $\mathbb{Z}_2$-vector space. The complex K is called **realizable modulo 2 by a bilinear form** $I\colon V\times V\to\mathbb{Z}_2$ **and a collection of vectors** y_σ **indexed by** k**-faces of** K if there is a general position PL map $f: K\to\mathbb{R}^{2k}$ such that

$$(R_{2,f})\qquad I(y_\sigma,y_\tau)=|f\sigma\cap f\tau|_2\quad\text{for any non-adjacent } k\text{-faces } \sigma,\tau \text{ of } K.$$

Lemma 2.1.1. *Let $I\colon V\times V\to\mathbb{Z}_2$ be an even/odd symmetric bilinear form on a $\mathbb{Z}_2$-vector space V. The complex K is realizable modulo 2 by I and some collection of vectors y_σ if and only if K is compatible modulo 2 to an even/odd matrix A such that $\operatorname{rk}A\le\operatorname{rk}I$.*

Lemma 2.1.1 implicitly appeared in [Bi21, §2]. Lemma 2.1.1 is easily deduced below from well-known Lemmas 2.1.2, 2.1.4, known Lemma 2.1.5 (see proof e.g. in [Al, Theorem 3] and [MW69, Theorem 1]) and simple Lemma 2.1.3 which appeared in a discussion with A. Bikeev and which is proved in [Bi21, §2].

Lemma 2.1.2. *Let F be $\mathbb{Z}_2$ or $\mathbb{Q}$. Let $I\colon V\times V\to F$ be a symmetric bilinear form on an F-vector space V. If A is a Gramian matrix with respect to I, then $\operatorname{rk}A\le\operatorname{rk}I$.*

Proof of the implication $(\Rightarrow)$ of Lemma 2.1.1 for I even. Denote by A the Gramian matrix of the vectors y_σ with respect to I. Then A is even and $(C_{2,f})$ holds. By Lemma 2.1.2 we have $\operatorname{rk}A\le\operatorname{rk}I$. ☐

Lemma 2.1.3. *Let $I\colon V\times V\to\mathbb{Z}_2$ be an odd symmetric bilinear form on a $\mathbb{Z}_2$-vector space V. If A is an even Gramian matrix with respect to I, then $\operatorname{rk}A\le\operatorname{rk}I-1$.*

Proof of the implication $(\Rightarrow)$ of Lemma 2.1.1 for an I odd. Denote by A the Gramian matrix of the vectors y_σ with respect to I.

If A is odd, then by Lemma 2.1.2 $\operatorname{rk}A\le\operatorname{rk}I$ and $(C_{2,f})$ holds.

If A is even, then by Lemma 2.1.3 $\operatorname{rk}A\le\operatorname{rk}I-1$. Take the matrix A' obtained from A by replacing the entry $A_{1,1}$ by 1. Then $\operatorname{rk}A'\le\operatorname{rk}I$, A' is odd and $(C_{2,f})$ holds. ☐

Lemma 2.1.4. *Let A,B be $n\times m$ and $m\times k$ matrices with entries in $\mathbb{Z}_2$, respectively. Then $\operatorname{rk}AB\le\operatorname{rk}B$. In particular, $\operatorname{rk}AB\le m$.*

Denote by H_2 the $2g\times 2g$-matrix with $\mathbb{Z}_2$-entries formed by g diagonal blocks $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and zeros elsewhere.

Recall that the rank of an even symmetric matrix with entries in $\mathbb{Z}_2$ is even.

Lemma 2.1.5. *Let A be a symmetric $a\times a$-matrix with $\mathbb{Z}_2$-entries. Let H be the matrix $H_{\frac{\operatorname{rk}A}{2}}$ if A is even and the $\operatorname{rk}A\times\operatorname{rk}A$ identity matrix otherwise. There is a $\operatorname{rk}A\times a$-matrix Y such that $A=Y^THY$.*

Proof of the implication $(\Leftarrow)$ of Lemma 2.1.1. Denote by I the matrix of the bilinear form I in some basis (e_i) of V. Denote $m:=\dim V$. For $M=A,I$ let H_M be the matrix $H_{\frac{\operatorname{rk}M}{2}}$ if M is even and the $\operatorname{rk}M\times\operatorname{rk}M$ identity matrix if M is odd. By Lemma 2.1.5 for each $M=A,I$ there is a matrix Y_M such that $M=Y_M^TH_MY_M$. By Lemma 2.1.4 we have $\operatorname{rk}Y_I\ge\operatorname{rk}H_IY_I\ge\operatorname{rk}I$. Hence all the $\operatorname{rk}I$ rows of Y_I are linearly independent.

Thus there is a nondegenerate $m \times m$-matrix Y'_I whose first $\operatorname{rk} I$ rows are the rows of Y_I. Denote by H'_I the $m \times m$-matrix $\begin{pmatrix} H_I & 0 \\ 0 & 0 \end{pmatrix}$. Then $I = (Y'_I)^T H'_I Y'_I$. Denote by Y the $m \times n$-matrix obtained from Y_A by adding $m - \operatorname{rk} A$ zeroes below each column of Y_A. Then $A = Y^T H'_I Y = ((Y'_I)^{-1}Y)^T I((Y'_I)^{-1}Y)$. Denote by $y_\sigma \in V$ the vector represented in the basis (e_i) by the corresponding column of the matrix $(Y'_I)^{-1}Y$. Then $(R_{2,f})$ holds. □

Denote by $H_{g,\mathbb{Z}}$ the $2g \times 2g$-matrix with $\mathbb{Z}$-entries formed by g diagonal blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and zeros elsewhere.

Lemma 2.1.6. *For every skew-symmetric $a \times a$-matrix A with $\mathbb{Z}$-entries its rank r is even and there is an $r \times a$ matrix Y with $\mathbb{Z}$-entries such that $A = Y^T H_{r/2,\mathbb{Z}} Y$.*

Lemma 2.1.6 follows from [Bo81, Chapter IX, §5, Theorem 1].

Lemma 2.1.7. *If a symmetric $n \times n$ matrix A with $\mathbb{Z}$-entries has rank 1 and any diagonal element of A is the square of an integer*[5]*, then $A = b^T b$ for some vector $b \in \mathbb{Z}^n$.*

Proof. Since A is symmetric and has rank 1, there are a nonzero vector $u \in \mathbb{Q}^n$ and a number $q \in \mathbb{Q}$ such that $A = qu^T u$. Then for each $i = 1, \dots, n$ we have $qu_i^2 = A_{i,i}$. Since $u \neq 0$, q is the square of a rational number. Hence for each $i = 1, \dots, n$ the number $\sqrt{q}u_i$ is an integer. Then let $b = \sqrt{q}u$. □

2.2 Construction of $\mathbb{Z}_2$- or $\mathbb{Z}$-embeddings

Lemma 2.2.1. *Let M be a $(k-1)$-connected PL $2k$-manifold, possibly with boundary. Then any homology class in $H_k(M)$ or in $H_k(M;\mathbb{Z})$ is represented by a general position PL map $S^k \to M$. For $k \geq 3$ this map can be assumed to be an embedding.*

A well-known proof. For $k = 1$ the lemma is obvious, so assume that $k \geq 2$. Then M is simply-connected. Hence by Hurewicz theorem $H_{k-1}(M;\mathbb{Z}) = 0$. So by the coefficient exact sequence the reduction modulo 2 $H_k(M;\mathbb{Z}) \to H_k(M)$ is epimorphic. Hence it suffices to prove the lemma for $\mathbb{Z}$.

By Hurewicz theorem the map $\pi_k(M) \to H_k(M;\mathbb{Z})$ is epimorphic. Hence any homology class $\alpha \in H_k(M;\mathbb{Z})$ is represented by a continuous map $S^k \to M$. Then by a small shift we obtain a general position PL map $S^k \to M$ representing α.

For $k \geq 3$ the latter map is homotopic to an embedding by the Penrose-Whitehead-Zeeman-Irwin Theorem [Ir65], cf. [Sk06, Theorem 2.9]. □

Proof of the implication ($\Leftarrow$) of Theorem 1.2.3. Since K is compatible to the matrix A, by (the implication ($\Leftarrow$) of) Lemma 2.1.1 K is realizable modulo 2 by $\cap_M$ and a collection $y_\sigma \in H_k(M)$. So there is a general position map $f : K \to \mathbb{R}^{2k}$ such that (R_f) holds for $I = \cap_M$.

Take a $2k$-ball $B \subset \operatorname{Int} M$. We may assume that $fK \subset B$. By Lemma 2.2.1 for any k-face σ the class y_σ is represented by a general position PL map $\widetilde{\sigma} : S^k \to M - B$. By general position $\widetilde{\sigma}(S^k) \cap B = \emptyset$. We may assume that the maps $\widetilde{\sigma}$ for different σ are in general position to each other. Take a general position PL map $h : K \to M$ obtained from f by connected summation of $f|_\sigma$ and $\widetilde{\sigma}$ along an arc l_σ, for every k-face σ.

[5] The last requirement can be replaced with "some nonzero diagonal element of A is the square of an integer".

(*) For $k \geq 2$ by general position we may assume that $l_\sigma \cap (f\tau \cup \widetilde{\tau}S^k \cup l_\tau) = \emptyset$ for $\sigma \neq \tau$. For $k = 1$ we may assume that for the tube U_σ along which we make connected summation for σ, each of the intersections of U_σ with $f\tau$, with $\widetilde{\tau}S^k$, and with any other tube, consists of an even number of points.

Now the map h is a $\mathbb{Z}_2$-embedding because for any non-adjacent k-faces σ, τ we have

$$|h\sigma \cap h\tau|_2 \overset{(1)}{=} |f\sigma \cap f\tau|_2 + |\widetilde{\sigma}S^k \cap \widetilde{\tau}S^k|_2 \overset{(2)}{=} |f\sigma \cap f\tau|_2 + y_\sigma \cap_M y_\tau \overset{(3)}{=} 0.$$

Here

- equality (1) holds because (*) holds, $\widetilde{\sigma}S^k \cap f\tau \subset \widetilde{\sigma}S^k \cap B = \emptyset$ and analogously $\widetilde{\tau}S^k \cap f\sigma = \emptyset$;
- equality (2) holds because $\widetilde{\sigma}S^k, \widetilde{\tau}S^k$ represent y_σ, y_τ, respectively;
- equality (3) is (R_f) for $I = \cap_M$. □

Proof of the implication ($\Leftarrow$) *of Theorem 1.3.4.* The proof is obtained from the proof of the implication ($\Leftarrow$) of Theorem 1.2.3 by the following changes. Replace the first paragraph by the following sentence.

Since K is realizable by $\cap_{M,\mathbb{Z}}$ and a collection $y_\sigma \in H_k(M)$, there are a general position map $f : K \to \mathbb{R}^{2k}$ and orientations on k-faces of K such that $(R_{f,\mathbb{Z}})$ holds for $I = \cap_{M,\mathbb{Z}}$.

Replace $\mathbb{Z}_2$ by $\mathbb{Z}$, 'an even number of points' by 'some points the sum of whose signs is zero', $|A \cap B|_2$ by $A \cdot B$, and $\cap_M$ by $\cap_{M,\mathbb{Z}}$. □

2.3 Deduction of realizability from $\mathbb{Z}_2$- or $\mathbb{Z}$-embeddability

The implication ($\Rightarrow$) *of Theorem 1.2.3* follows by (the implication ($\Rightarrow$) of) Lemma 2.1.1 and Lemma 2.3.1.

Lemma 2.3.1. *If M is $(k-1)$-connected and there is a $\mathbb{Z}_2$-embedding $h : K \to M$, then K is realizable modulo 2 by $\cap_M$ and a collection of homology classes $y_\sigma \in H_k(M)$.*[6]

For a proof we need the following essentially known definitions and lemma.

In the rest of this subsection M is a compact connected PL $2k$-manifold (not necessarily closed or $(k-1)$-connected), and $g : K \to M$ is any general position PL map.

Take any pair of non-adjacent k-faces σ, τ. By general position the intersection $g\sigma \cap g\tau$ consists of a finite number of points. Assign to the pair $\{\sigma, \tau\}$ the residue

$$\nu(g)\{\sigma, \tau\} := |g\sigma \cap g\tau|_2.$$

Denote by K^* the set of all unordered pairs of non-adjacent k-faces of K. The obtained map $\nu(g) : K^* \to \mathbb{Z}_2$ is called the (modulo 2) **intersection cocycle** of g (we call it 'cocycle' instead of 'map' to avoid confusion with maps to M). Maps $K^* \to \mathbb{Z}_2$ are identified with subsets of K^* consisting of pairs going to $1 \in \mathbb{Z}_2$. (Maps $K^* \to \mathbb{Z}_2$ can also be regarded as 'partial matrices', i.e., symmetric arrangements of zeroes and ones in those cells of the $n \times n$-matrix that correspond to the pairs of non-adjacent k-faces.)

Let α be a $(k-1)$-face of K which is not contained in the boundary of a k-face σ of K. An **elementary coboundary** of the pair (α, σ) is the subset $\delta_K(\alpha, \sigma) \subset K^*$ consisting of all pairs $\{\sigma, \tau\}$ with $\tau \supset \alpha$.

[6]The proof shows that the $(k-1)$-connectedness assumption can be weakened to 'every map $K^{(k-1)} \to M$ extendable to K is null-homotopic' (or to the equivalent more cumbersome condition (H) of [PT19, §1.1]).

Cocycles $\nu, \nu' : K^* \to \mathbb{Z}_2$ (or $\nu, \nu' \subset K^*$) are called **cohomologous** (modulo 2) if

$$\nu - \nu' = \delta_K(\alpha_1, \sigma_1) + \ldots + \delta_K(\alpha_k, \sigma_k)$$

for some $(k-1)$-faces $\alpha_1, \ldots, \alpha_k$ and k-faces $\sigma_1, \ldots, \sigma_k$ (not necessarily distinct). Here $+$ is the componentwise addition (corresponding to the sum modulo 2 of subsets of K^*).

Lemma 2.3.2. *Let $\nu : K^* \to \mathbb{Z}_2$ be a map. There is a general position PL map $g' : K \to M$ homotopic to g and such that $\nu(g') = \nu$ if and only if ν is cohomologous to $\nu(g)$.*

Remark 2.3.3. Lemmas 2.3.2 and 2.3.4 are essentially known. They were proved by van Kampen and Shapiro for $M = \mathbb{R}^{2k}$, see a proof in [Sk18, Lemma 1.5.8 and Proposition 1.5.9] (modulo 2 version for $k = 1$) and in [Sh57, Lemma 3.5] [FKT, §2] (integer version for any k, cf. [Sk18, Remark 1.6.4]). The proof for an arbitrary M is analogous (e.g. the finger moves are done in a regular neighborhoods of a path in M, which is homeomorphic to the $2k$-ball). The 'only if' part of Lemma 2.3.4 is [Jo02, Theorem 1], [PT19, Lemma 11], the 'if' part of Lemma 2.3.4 is [Jo02, Theorem 6] (in which the assumptions '$n \geq 3$' and 'M is 1-connected' are superfluous).

Proof of Lemma 2.3.1. Denote by $K^{(k-1)}$ the union of all those faces of K whose dimension is less than k. Take a $2k$-ball $B \subset \operatorname{Int} M$. Since M is $(k-1)$-connected, h is homotopic to a map $h' : K \to M$ such that $h'(K^{(k-1)}) \subset \partial B$. By general position we may assume that $h'_{K^{(k-1)}}$ is an embedding into ∂B, and $K \cap \operatorname{Int} B = \emptyset$.

For any k-face σ let $\overline{\sigma}$ be the cone over $h'\partial\sigma$ with a vertex in $\operatorname{Int} B$. We may assume that these vertices are in general position. Define a map $f' : K \to B$ to be h' on $K^{(k-1)}$, and to be the cone map on any k-face σ, so that $f'\sigma = \overline{\sigma}$. Then f' is a general position PL map.

It remains to prove that the complex K is realizable by $\cap_M$ and the collection $y_\sigma := [f'\sigma \cup h'\sigma]$. By Lemma 2.3.2 the intersection cocycles $\nu(h) = 0$ and $\nu(h')$ are cohomologous. Hence by Lemma 2.3.2 for $M = B$ there is a general position PL map $f : K \to B$ homotopic to f' and such that

$$(*) \qquad \nu(f) = \nu(f') + \nu(h').$$

Now (R_f) follows because for any non-adjacent edges σ, τ we have

$$|f\sigma \cap f\tau|_2 \overset{(1)}{=} |f'\sigma \cap f'\tau|_2 + |h'\sigma \cap h'\tau|_2 \overset{(2)}{=} |(f'\sigma \cup h'\sigma) \cap (f'\tau \cup h'\tau)|_2 \overset{(3)}{=} y_\sigma \cap_M y_\tau.$$

Here

• equality (1) holds by (*) because $\nu(f)\{\sigma, \tau\} = |f\sigma \cap f\tau|_2$ and the same for f', h';

• equality (2) holds because $\sigma \cap \tau = \emptyset$ and $h'_{K^{(k-1)}}$ is an embedding, so that $f'\sigma \cap h' \operatorname{Int} \tau \subset B \cap h'\tau = \emptyset$, and analogously $f'\tau \cap h'\sigma = \emptyset$.

• equality (3) holds by general position and definitions of y_σ, y_τ and $\cap_M$. ☐

Proof of the implication (⇒) of Theorem 1.3.4. The proof is obtained from the proof of Lemma 2.3.1 by the following changes. Replace (R_f) by $(R_{f,\mathbb{Z}})$, ν by $\nu_{\mathbb{Z}}$, $|A \cap B|_2$ by $A \cdot B$, $\cap_M$ by $\cap_{M,\mathbb{Z}}$, and $f'\sigma \cup h'\sigma$ by the integer cycle formed by the union $f'\sigma \cup (-h'\sigma)$ of $f'\sigma$ with the orientation coming from the orientation of σ, and of $h'\sigma$ with the orientation coming from the opposite orientation of σ (and make the same replacement for τ instead of σ). Refer to Lemma 2.3.4 below instead of Lemma 2.3.2. ☐

Assume that M is oriented. Take any orientations on k-faces of K. Assign to any ordered pair (σ, τ) of non-adjacent k-faces the integer

$$\nu_{\mathbb{Z}}(g)(\sigma, \tau) := g\sigma \cdot g\tau.$$

Denote by $\widetilde{K}$ the set of all ordered pairs of non-adjacent k-faces of K. The obtained map $\nu_{\mathbb{Z}}(g) : \widetilde{K} \to \mathbb{Z}$ is called the (integer) **intersection cocycle** of f. This cocycle is **super-symmetric**, i.e., $\nu_{\mathbb{Z}}(g)(\sigma, \tau) = (-1)^k \nu_{\mathbb{Z}}(g)(\tau, \sigma)$.

For oriented edge AB of K denote $[AB : B] = 1$ and $[AB : A] = -1$ (AB *goes* to B and *issues* out of A). For oriented 2-face ABC of K denote $[ABC : AB] = [ABC : BC] = [ABC : CA] = 1$ and $[ABC : BA] = [ABC : CB] = [ABC : AC] = -1$ (this disagrees with [Sk18, §2.4.2], but both definitions work). Analogously, for oriented k-face $A_0A_1 \ldots A_k$ of K and the oriented $(k-1)$-face α obtained from $A_0A_1 \ldots A_k$ by dropping the vertex A_l denote $[A_0A_1 \ldots A_k : \alpha] = (-1)^l$. For an oriented $(k-1)$-face α' obtained from α by an odd permutation let $[A_0A_1 \ldots A_k : \alpha'] = -[A_0A_1 \ldots A_k : \alpha]$.

Let α be an oriented $(k-1)$-face of K which is not contained in the boundary of a k-face σ of K. An (integer) **elementary coboundary** of the pair (α, σ) is the map $\delta_K(\alpha, \sigma) : \widetilde{K} \to \mathbb{Z}$ assigning

$$(-1)^k[\tau : \alpha] \text{ to } (\sigma, \tau), \quad [\tau : \alpha] \text{ to } (\tau, \sigma) \quad \text{and} \quad 0 \text{ to any other pair.}$$

Cocycles $\nu, \nu' : \widetilde{K} \to \mathbb{Z}$ are called (integer) **cohomologous** if

$$\nu - \nu' = c_1\delta_K(\alpha_1, \sigma_1) + \ldots + c_k\delta_K(\alpha_k, \sigma_k)$$

for some integers $c_1, \ldots, c_k \in \mathbb{Z}$, $(k-1)$-faces $\alpha_1, \ldots, \alpha_k$, and k-faces $\sigma_1, \ldots, \sigma_k$ (not necessarily distinct). Observe that change of the orientation of α forces change of the sign of $\delta_K(\alpha, \sigma)$. Hence the cohomology equivalence relation does not depend on the orientations of $(k-1)$-faces.

Lemma 2.3.4. *Assume that M is oriented and $\nu : \widetilde{K} \to \mathbb{Z}$ is a super-symmetric map. There is a general position PL map $g' : K \to M$ homotopic to g and such that $\nu_{\mathbb{Z}}(g') = \nu$ if and only if ν is cohomologous to $\nu_{\mathbb{Z}}(g)$.*

2.4 Byproducts: additivity and even embeddings

The **$\mathbb{Z}_2$-rank** $r(X)$ of a k-complex X is the minimal number $\operatorname{rk} \cap_N$ such that N is a $(k-1)$-connected PL $2k$-manifold to which X has a $\mathbb{Z}_2$-embedding.

Theorem 2.4.1 (proved in §2.4). *We have $r(K \sqcup L) = r(K) + r(L)$ for any two k-complexes K, L.*

A map $f : K \to M$ is called **even** if $f_*\alpha \cap_M f_*\alpha = 0$ for any $\alpha \in H_k(K)$. For $k = 1$ this is equivalent to *orientability* of f: for any cycle subgraph $C \subset K$ the orientation of M does not change along fC.

Theorem 2.4.2. *If a k-complex has an even $\mathbb{Z}_2$-embedding to M and $\cap_M$ is odd, then the complex has a $\mathbb{Z}_2$-embedding to the connected sum of $\left\lfloor \dfrac{\operatorname{rk} \cap_M - 1}{2} \right\rfloor$ copies of $S^k \times S^k$.*

For proofs we need the following addendum to Lemma 2.3.1.

A *maximal k-forest* $T \subset K$ is a maximal subcomplex $T \subset K$ such that $H_k(T) = 0$. For any k-face $\sigma \subset T$ set $\widehat{\sigma} = \widehat{\sigma}_{K,T} := 0$. For any k-face $\sigma \subset K - T$ there is a unique $\widehat{\sigma} = \widehat{\sigma}_{K,T} \in H_k(K)$ coming from $H_k(T \cup \sigma)$. (For $k = 1$ we have that $\widehat{\sigma}$ is the union of σ and the simple path in T joining the ends of σ.)

Addendum 2.4.3. *In Lemma 2.3.1 for any maximal k-forest $T \subset K$ we can take the collection $y_\sigma = h_*\widehat{\sigma}$.*

Lemma 2.4.4. *If T is a k-complex and $H_k(T) = 0$, then any map of T to a $(k-1)$-connected space is homotopic to the map to a point.*

Sketch of a well-known proof. Since T is a k-complex and $H_k(T) = 0$, we have $H_k(T;\mathbb{Z}) = 0$. Now the lemma is proved either applying the relative Hurewicz theorem to the mapping cylinder of the map, or using obstruction theory. ☐

Proof of Addendum 2.4.3. Since M is $(k-1)$-connected, by Lemma 2.4.4 we can make a homotopy of h' and assume additionally that $h'(T)$ is contained in a small neighborhood of ∂B. Then

$$[f'\sigma \cup h'\sigma] = [f'\sigma \cup h'(\widehat{\sigma} - \sigma)] + [h'\widehat{\sigma}] = 0 + h'_*\widehat{\sigma} = h_*\widehat{\sigma}.$$

☐

Lemma 2.4.5. *Let M be a $(k-1)$-connected PL $2k$-manifold.*

(a) If $\cap_M$ is odd, then there is a $(k-1)$-connected PL $2k$-manifold N_1 such that $\operatorname{rk} \cap_{N_1} = 1$.

(b) Let A be the Gramian matrix of some homology classes in $H_k(M)$ with respect to $\cap_M$. Then there is a $(k-1)$-connected PL $2k$-manifold N such that $\cap_N$ has the same rank and type as A.

Proof. (a)[7] If $k = 1, 2$, then $N_1 = \mathbb{R}P^2, \mathbb{C}P^2$ satisfy the requirements.

Assume that $k \geq 3$. Since $\cap_M$ is odd, there is $\alpha \in H_k(M)$ such that $\alpha \cap_M \alpha = 1$. By Lemma 2.2.1 α is represented by an embedding $f\colon S^k \to M$. Let N_1 be the regular neighbourhood of $f(S^k)$. Then N_1 is $(k-1)$-connected and $\operatorname{rk} \cap_{N_1} \leq \operatorname{rk} H_k(N_1) = \operatorname{rk} H_k(S^k) = 1$. Since $\alpha \cap_M \alpha = 1$, it follows that $\operatorname{rk} \cap_{N_1} = 1$.

(b) If A is even, then $\operatorname{rk} A$ is even. So let N be the connected sum of $\operatorname{rk} A/2$ copies of $S^k \times S^k$.

If A is odd, then $\cap_M$ is odd. Hence by (a) there is a manifold N_1 such that $\operatorname{rk} \cap_{N_1} = 1$. So let N be the connected sum of $\operatorname{rk} A$ copies of N_1.

In both cases $\cap_N$ has the same rank and type as A. ☐

Proof of Theorem 2.4.1. The inequality $r(K \sqcup L) \leq r(K) + r(L)$ is clear. Let us prove that $r(K \sqcup L) \geq r(K) + r(L)$. It suffices to prove that $\operatorname{rk} \cap_M \geq r(K) + r(L)$ if there is a PL $\mathbb{Z}_2$-embedding $h : K \sqcup L \to M$.

Let T be a maximal forest of the complex $K \sqcup L$. Then $T \cap K, T \cap L$ are the maximal forests of K, L, respectively. Denote by $A_{\sigma,\tau} := h_*\widehat{\sigma} \cap_M h_*\widehat{\tau}$ the Gramian matrix with respect to $\cap_M$ of the homology classes $h_*\widehat{\sigma}$ corresponding to k-faces σ of $K \sqcup L$. Denote by A_K, A_L the Gramian matrices of the homology classes $h_*\widehat{\sigma}$ corresponding to k-faces σ of K and of L, respectively. Then A_K, A_L are submatrices of A. So

$$\operatorname{rk} \cap_M \overset{(1)}{\geq} \operatorname{rk} A \overset{(2)}{=} \operatorname{rk} A_K + \operatorname{rk} A_L \overset{(3)}{\geq} r(K) + r(L).$$

[7]Part (a) follows from the PL analogue mentioned in Remark 1.2.4.a. We present a much simpler direct proof.

Here

- (1) follows by Lemma 2.1.2;
- (2) follows because for any $\sigma \in K, \tau \in L$ we have

$$A_{\sigma,\tau} = h_*\widehat{\sigma} \cap_M h_*\widehat{\tau} = \sum_{\sigma' \in C_\sigma, \tau' \in C_\tau} |h\sigma' \cap h\tau'|_2 = 0$$

where C_σ, C_τ are some k-cycles representing $\widehat{\sigma}, \widehat{\tau}$, and the last equality holds because h is a $\mathbb{Z}_2$-embedding.

The inequality (3) follows from the inequalities $\operatorname{rk} A_K \geq r(K)$ and $\operatorname{rk} A_L \geq r(L)$. In the following paragraph we prove that $\operatorname{rk} A_K \geq r(K)$. The inequality $\operatorname{rk} A_L \geq r(L)$ is proved analogously.

By Addendum 2.4.3 the complex K is realizable by $\cap_M$ and the homology classes $h_*\widehat{\sigma}$ corresponding to k-faces σ of K. Hence K is compatible to A_K (cf. Lemma 2.1.1). By Lemma 2.4.5.b there is a $(k-1)$-connected PL $2k$-manifold N such that $\cap_N$ has the same rank and type as A_K. By Theorem 1.2.3 there is a $\mathbb{Z}_2$-embedding of K to N. Then $\operatorname{rk} A_K = \operatorname{rk} \cap_N \geq r(K)$. □

Comment. Analogues of Theorem 2.4.1 hold for *even*, for *odd* and presumably for *closed* $\mathbb{Z}_2$-ranks, which are defined analogously. The $\mathbb{Z}$-*rank* $r_{\mathbb{Z}}(X)$ of a complex X is defined to be the minimal number $\operatorname{rk} \cap_{N,\mathbb{Z}}$ such that N is a $(k-1)$-connected PL $2k$-manifold to which X has a $\mathbb{Z}$-embedding. We conjecture that $r_{\mathbb{Z}}(K \sqcup L) = r_{\mathbb{Z}}(K) + r_{\mathbb{Z}}(L)$ for any two k-complexes K, L.

Proof of Theorem 2.4.2. Take an even $\mathbb{Z}_2$-embedding $h : K \to M$ and a maximal k-forest in K. Since $\cap_M$ is odd, by Lemma 2.1.5 there is a basis $H_k(M)$ in which the matrix I of $\cap_M$ is diagonal, with only units and zeroes on the diagonal. Let Y be the matrix formed by the coordinates Y_σ of homology classes $y_\sigma := h_*\widehat{\sigma}$ in this basis. Denote $A := Y^T I Y$.

Since h is even, we have $A_{\sigma,\sigma} = Y_\sigma^T I Y_\sigma = h_*\widehat{\sigma} \cap_M h_*\widehat{\sigma} = 0$, so A is even.

We have $I = I_1^T I_1$, where I_1 is a $\operatorname{rk} \cap_M \times \operatorname{rk} H_k(M)$ diagonal matrix with units on the diagonal. Then $A = Y^T I_1^T I_1 Y = (I_1 Y)^T I_1 Y$. Hence by Lemma 2.1.3 we have $\operatorname{rk} A \leq \operatorname{rk} \cap_M - 1$. Since A is even, $\operatorname{rk} A$ is even. Hence $\operatorname{rk} A \leq 2g := 2\left\lfloor \dfrac{\operatorname{rk} \cap_M - 1}{2} \right\rfloor$.

Thus by Lemma 2.3.1, Addendum 2.4.3 and Lemma 2.1.1 K is compatible to A. So by Theorem 1.2.3 K has a $\mathbb{Z}_2$-embedding to the connected sum of g copies of $S^k \times S^k$. □

Remark 2.4.6. The following are equivalent:

(EV) there is an even $\mathbb{Z}_2$-embedding $K \to M$;

(CV) K is compatible modulo 2 to an even matrix A with $\mathbb{Z}_2$-entries such that $\operatorname{rk} A \leq \operatorname{rk} \cap_M$;

(RV) K is realizable modulo 2 by $\cap_M$ and a collection of homology classes $y_\sigma \in H_k(M)$ such that $y_\sigma \cap_M y_\sigma = 0$ for each k-face σ;

Proof. The equivalence (CV)⇒(RV) is proved analogously to Lemma 2.1.1.

The implication (RV)⇒(EV) is proved analogously to the implication (⇐) of Theorem 1.2.3 (§2.2). The obtained $\mathbb{Z}_2$-embedding $h : K \to M$ is even because for any k-cycle C modulo 2 in K we have

$$h_*[C] = \sum_{\sigma \in C} f\sigma \# \widetilde{\sigma} S^k = \sum_{\sigma \in C} f\sigma + \sum_{\sigma \in C} \widetilde{\sigma} S^k = \sum_{\sigma \in C} \widetilde{\sigma} S^k = \sum_{\sigma \in C} y_\sigma,$$

which together with $y_\sigma \cap_M y_\sigma = 0$ implies that $h_*[C] \cap_M h_*[C] = 0$.

Let us prove the implication (EV)⇒(RV). Let $h\colon K \to M$ be an even $\mathbb{Z}_2$-embedding. Take a maximal forest in K. By Addendum 2.4.3 we can take the collection $y_\sigma = h_*\widehat{\sigma}$. Then $y_\sigma \cap_M y_\sigma = h_*\widehat{\sigma} \cap_M h_*\widehat{\sigma} = 0$.

2.5 Appendix: reformulations of the criteria

Proposition 2.5.1. *The following are equivalent:*

(E) there is a $\mathbb{Z}_2$-embedding $K \to M$;

(EH) there is a cohomological $\mathbb{Z}_2$-embedding *$K \to M$, i.e., a general position PL map $h : K \to M$ such that $\nu(h)$ is cohomologous to zero;*

(C) K is compatible modulo 2 to a matrix A such that $\operatorname{rk} A \le \operatorname{rk} \cap_M$ and A is even/odd if $\cap_M$ is even/odd;

(CM) K is compatible modulo 2 to a matrix A such that $\operatorname{rk} A \le \operatorname{rk} \cap_M$ and A is $\begin{pmatrix} H_g & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}$ if $\cap_M$ is even or odd, respectively.

(R) K is realizable modulo 2 by $\cap_M$ and a collection of homology classes $y_\sigma \in H_k(M)$;

(R') there is a collection of homology classes $y_\sigma \in H_k(M)$ such that for any $2k$-cycle $C \subset K^*$ we have $\sum\limits_{\{\sigma,\tau\}\in C} y_\sigma \cap_M y_\tau = v_C(K)$.

(RT) there is a homomorphism $\psi : H_k(K) \to H_k(M)$ such that for any $2k$-cycle $C \subset K^*$ and for some (equivalently, for any) maximal k-forest in K we have $\sum\limits_{\{\sigma,\tau\}\in C} \psi\widehat{\sigma} \cap_M \psi\widehat{\tau} = v_C(K)$.

(RI) there is a homomorphism $\psi : H_k(K) \to H_k(M)$ such that if $\alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_q$ are k-cycles in K and $C := \sum\limits_{j=1}^{q} [\alpha_j \times \beta_j]$ is a $2k$-cycle in K^*, then[8] *$\sum\limits_{j=1}^{q} \psi\alpha_j \cap_M \psi\beta_j = v_C(K)$.*

(RH) there is a homomorphism $\psi : H_k(K) \to H_k(M)$ such that $\omega(\psi) = v(K)$.

The set K^*, the intersection cocycle $\nu(f)$ and cohomology of cocycles are defined in §2.3. By Lemma 2.3.2 the property of being a cohomological $\mathbb{Z}_2$-embedding is homotopy invariant (as opposed to the property of being a $\mathbb{Z}_2$-embedding).

Definitions required for (R'). A subset $C \subset K^*$ is called a *$2k$-cycle* (modulo 2) if for each k-face σ and $(k-1)$-face a there is an even number of k-faces $\tau \supset a$ such that $\{\sigma, \tau\} \in C$.

For a $2k$-cycle $C \subset K^*$ and a general position PL map $f : K \to \mathbb{R}^{2k}$ the *C-van Kampen number* is defined as $v_C(K) := \sum\limits_{\{\sigma,\tau\}\in C} |f\sigma \cap f\tau|_2$. This depends only on K, C, not on f.

Cf. [Sk18, Remark 1.5.5.c].

Comment. Clearly,

(a) for any disjoint k-cycles α, β in K considered as unions of k-faces the following is a $2k$-cycle:

$$[\alpha \times \beta] := \{\{\sigma, \tau\} \in K^* \ : \ \sigma \subset \alpha,\ \tau \subset \beta\};$$

(b) $(K_5)^*$ and $(K_{3,3})^*$ are 2-cycles;

(c) the sum (modulo 2) of $2k$-cycles is a $2k$-cycle.

[8] This C is automatically a $2k$-cycle in $K^2/\mathbb{Z}_2 \supset K^*$. Since K is k-dimensional, $H_k(K)$ is the group of k-cycles in K, so $\psi\alpha_j$ and $\psi\beta_j$ do make sense.

For $k = 1$ any 2-cycle in K^* is a sum of 2-cycles from (a) and (b) above [Sa91, §3.4].

Definition required for (RI). For k-cycles α, β in K considered as unions of k-faces let

$$[\alpha \times \beta] := \{\{\sigma, \tau\} \in K^* \ : \ \sigma \subset \alpha, \ \tau \subset \beta, \ \{\sigma, \tau\} \not\subset \alpha \cap \beta\}.$$

This is a $2k$-cycle.

Definitions required for (RH). For an assignment $\overline{\psi}$ of elements of $H_k(M)$ on k-faces of K define a mapping (a cocycle)

$$\omega(\overline{\psi}) : K^* \to \mathbb{Z}_2 \quad \text{by} \quad \omega(\overline{\psi})(\sigma, \tau) := \overline{\psi}\sigma \cap_M \overline{\psi}\tau.$$

For any homomorphism $\psi : H_k(K) \to H_k(M)$ there is an assignment $\overline{\psi}$ of elements of $H_k(M)$ on k-faces of K such that $\psi(\sigma_1 + \ldots + \sigma_s) = \overline{\psi}(\sigma_1) + \ldots + \overline{\psi}(\sigma_s)$ for any k-cycle $\sigma_1 + \ldots + \sigma_s$. (In other words, ψ can be regarded as an element of $H^k(K; H_k(M))$ which is such an assignment $\overline{\psi}$ up to cohomology.) Let $H^{2k}(K^*)$ be the group of modulo 2 cohomology classes of cocycles $K^* \to \mathbb{Z}_2$. It is easy to check that

$$\omega(\psi) := [\omega(\overline{\psi})] \in H^{2k}(K^*)$$

is well-defined, i.e. that cocycles $\omega(\overline{\psi})$ are cohomologous for cohomologous assignments $\overline{\psi}$.

By Lemma 2.3.2 for $M = \mathbb{R}^{2k}$ the class $v(K) := [\nu(f)] \in H^{2k}(K^*)$, where $f : K \to \mathbb{R}^2$ is any general position map, is well-defined.

Remark 2.5.2. (a) The properties (RH) and (RI) are topologically invariant, i.e., are stated in terms of group $H_k(K)$ and elements of the group $H^{2k}(K^*)$, which groups are isomorphic for PL homeomorphic complexes K, and which elements map naturally under PL homeomorphisms. In spite of that, the property (RH) is similar to the non-invariant statement of [PT19, Theorem 1].

(b) An alternative way to state the above definition of $\omega(\psi)$ (analogous to [Kr00, before Theorem 3.2]) is $p^*\omega(\psi) = ((\psi \times \psi)^* \cap_M)|_{\widetilde{K}}$, where $p : \widetilde{K} \to K^*$ is the quotient map and $\cap_M$ is considered as an element of $\mathrm{Hom}(H_k(M) \otimes H_k(M)) \cong H^k(M) \otimes H^k(M)$ which is a direct summand of $H^{2k}(M \times M)$ by the Künneth formula. Since $\cap_M$ is symmetric, $((\psi \times \psi)^* \cap_M)|_{\widetilde{K}}$ is symmetric, hence this restriction comes from $H^{2k}(K^*)$.

Below (except for the proof of Proposition 2.5.1) M need not be closed or $(k-1)$-connected.

A map $g : K \to M$ *induces* the homomorphism $g_* : H_k(K) \to H_k(M)$ by the formula $g_*[\alpha] := [g\alpha]$ for any $[\alpha] \in H_k(K)$. The following lemma is simple and well-known.

Lemma 2.5.3. *Any homomorphism $H_k(K) \to H_k(M)$ is induced by some map $g : K \to M$.*

Proposition 2.5.4. *Let T be a maximal k-forest of K.*

(a) Every k-cycle α in K equals to $\sum\limits_{\sigma \in \alpha} \widehat{\sigma}$.

(b) Then $\overline{T} := T \times K \cup K \times T$ is a maximal $2k$-forest of $K \times K$.[9]

(c) For this maximal k-forest and any k-faces σ, τ of K we have $\widehat{\sigma \times \tau} = \widehat{\sigma} \times \widehat{\tau}$.

(d) For any $2k$-cycle Z in the simplicial deleted product

$$\widetilde{K} := \cup\{\sigma \times \tau \subset |K| \times |K| \ : \ \sigma \cap \tau = \emptyset\}$$

[9]The product $K \times K$ has a natural structure of a cell complex, the cells being $\sigma \times \tau$ for faces σ, τ of K. A maximal k-forest and $\widehat{\sigma \times \tau}$ are defined for cell complexes analogously to the simplicial complexes.

symmetric w.r.t. the involution $(x,y)\leftrightarrow(y,x)$ we have $Z=\sum\limits_{\{\sigma,\tau\}:\ (\sigma,\tau)\in Z}(\widehat\sigma\times\widehat\tau+\widehat\tau\times\widehat\sigma)$.

(e) For any $2k$-cycle $C\subset K^*$ we have $C=\sum\limits_{\{\sigma,\tau\}\in C}[\widehat\sigma\times\widehat\tau]$.

Proof. Part (a) follows because the difference is a k-cycle, lies in T, and hence is trivial.

Proof of (b,c). Using the Mayer-Vietoris sequence and Künneth formula, we see that $H_{2k}(\overline{T})=0$.[10] For any pair $\sigma\times\tau\in K\times K$ we have a nontrivial $2k$-cycle $\widehat\sigma\times\widehat\tau\subset\overline{T}\cup\{\sigma\times\tau\}$.

Part (d) follows because

$$Z\overset{(1)}{=}\sum_{\sigma\times\tau\in Z}\widehat{\sigma\times\tau}\overset{(2)}{=}\sum_{\sigma\times\tau\in Z}\widehat\sigma\times\widehat\tau\overset{(3)}{=}\sum_{\{\sigma,\tau\}:\ (\sigma,\tau)\in Z}(\widehat\sigma\times\widehat\tau+\widehat\tau\times\widehat\sigma).$$

Here (1) holds by (a), (2) holds by (c), and (3) holds because $Z\in\widetilde{K}$ and Z is symmetric.

Part (e) follows by (d) because the $\mathbb{Z}_2$-quotient of $\widehat\sigma\times\widehat\tau+\widehat\tau\times\widehat\sigma$ is $[\widehat\sigma\times\widehat\tau]$. □

Comment. Propositions 2.5.4.de describe simple generators of $H_{2k}(K^*),H_{2k}(\widetilde{K})$ which are not elements of $H_{2k}(K^*),H_{2k}(\widetilde{K})$. This is useful for our applications, perhaps could be useful elsewhere, but properly considered to be not very natural. For this reason Propositions 2.5.4.de can be new, although homology of the deleted product have been much studied, for complexes see e.g. [FH10, MS17] and the references therein.

Lemma 2.5.5. *Let $g:K\to M$ be a map whose restriction to the $(k-1)$-skeleton $K^{(k-1)}$ of K is null-homotopic. Take a $2k$-ball $B\subset M$ and a general position map $g':K\to M$ homotopic to g such that $g'(K)\subset M-\operatorname{Int}B$ and $g'(K^{(k-1)})\subset\partial B$. For each k-face σ of K take a k-disk $\overline\sigma\subset B$ spanning $g'(\partial\sigma)$. For any non-adjacent k-faces σ,τ let*

$$\omega(g)(\sigma,\tau):=\Big|(\overline\sigma\cup g'\sigma)\bigcap(\overline\tau\cup g'\tau)\Big|_2.$$

(a) The cocycle $\omega(g)$ is well-defined, i.e. is independent on B,g' for fixed g.

(b) The cohomology class of $\omega(g)$ is $\omega(g_*)$.

(c) The cohomology class of $\nu(g)$ is $\omega(g_*)-v(K)$.

Lemma 2.5.5 (and its integer version of Remark 2.5.6.d) is essentially proved in [PT19, Theorem 13]. Parts (a,b) are simple. Part (c) follows from

$$\omega(g)(\sigma,\tau)=|g'\sigma\cap g'\tau|_2+|\overline\sigma\cap\overline\tau|_2=(\nu(g)+\nu(f_0))(\sigma,\tau),$$

where $f_0:K\to B$ is the map coinciding with g' on $K^{(k-1)}$ and such that $f_0(\sigma)=\overline\sigma$ for any k-face σ.

Proof of Proposition 2.5.1. The equivalence (E)⇔(C) forms Theorem 1.2.3.

The implication (E)⇒(EH) is clear. The implication (E)⇐(EH) is the particular case of Lemmas 2.3.2 for $\nu=0$.

We have (C)⇔(R) by Lemma 2.1.1, and (C)⇔(CM) by Remark 1.2.4.a.

The implication (E)⇒(R) is Lemma 2.3.1. The implication (R)⇒(E) is essentially proved in the proof of (⇐) of Theorem 1.2.3 (§2.2).

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[10] Here is an independent proof. Take a $2k$-cycle $Z\subset\overline{T}$.
If $Z\supset T\times T$, then $Z=\emptyset$ because by Künneth formula $H_{2k}(T\times T)=H_k(T)\otimes H_k(T)=0$.
If $Z\not\supset T\times T$, then w.l.o.g. for some k-face $\sigma\in K-T$ the set $Z_\sigma:=\{\tau\ :\ (\tau,\sigma)\in Z\}$ is non-empty. This set is a k-cycle in T. A contradiction.

The implications (R)⇒(R')⇒(RI) are clear.

The implication (RH)⇒(EH) follows from Lemmas 2.5.3 and 2.5.5.

So it remains to prove the implications (RI)⇒(RT)(any) and (RT)(some)⇒(R')⇒(RH).

The implication (RT)(some)⇒(R') follows because we can take $y_\sigma = \psi\widehat{\sigma}$.

In this paragraph we prove the implication (R')⇒(RH). Define the homomorphism $\psi\colon H_k(K) \to H_k(M)$ by the formula $\psi[C] = \sum\limits_{\sigma\in C} y_\sigma$. Take a general position map $f\colon K \to \mathbb{R}^{2k}$. We have $\omega(\psi) = [\omega(\overline{\psi})] = [\nu(f)] = v(K)$. Here the first and the third equalities are obvious. The second equality holds because the scalar product $H^{2k}(K^*)\times H_{2k}(K^*) \to \mathbb{Z}_2$ is nondegenerate and for any $2k$-cycle $C \subset K^*$ we have $\sum\limits_{\{\sigma,\tau\}\in C} (\omega(\psi) - \nu(f))(\sigma,\tau) = 0$.

In this paragraph we prove the implication (RI)⇒(RT)(any). Denote $y_\sigma = \psi\widehat{\sigma}$. By Proposition 2.5.4.e there are k-cycles $\alpha_j, \beta_j, j = 1, \ldots, q$ such that $C = \sum\limits_{j=1}^{q}[\alpha_j \times \beta_j]$. Then

$$\sum_{\{\sigma,\tau\}\in C} y_\sigma \cap_M y_\tau \overset{(1)}{=} \sum_{j=1}^{q} \sum_{\{\sigma,\tau\}\in[\alpha_j\times\beta_j]} y_\sigma \cap_M y_\tau \overset{(2)}{=}$$

$$\overset{(2)}{=} \sum_{j=1}^{q} \sum_{\sigma\in\alpha_j,\tau\in\beta_j} y_\sigma \cap_M y_\tau \overset{(3)}{=} \sum_{j=1}^{q} \psi\alpha_j \cap_M \psi\beta_j \overset{(4)}{=} v_C(K).$$

Here

- (1) follows by definition of C;
- (2) follows because for every pair $\{\sigma, \tau\} \subset \alpha_j \cap \beta_j$ we have $y_\sigma \cap_M y_\tau + y_\tau \cap_M y_\sigma = 0$;
- (3) follows by Proposition 2.5.4.a;
- (4) follows by (RI). □

Remark 2.5.6 (integer versions)**.** In this remark M is an oriented PL $2k$-manifold.

(a) Let $H^{2k}_s(\widetilde{K};\mathbb{Z})$ be the group of cohomology classes of super-symmetric cocycles $\widetilde{K} \to \mathbb{Z}$ (defined in §2.3). Analogously to the above one defines $v_{C,\mathbb{Z}}(K) \in \mathbb{Z}$, $\omega_{\mathbb{Z}}(\overline{\psi})$ and $\omega_{\mathbb{Z}}(\psi), v_{\mathbb{Z}}(K) \in H^{2k}_s(\widetilde{K};\mathbb{Z})$ for a homomorphism $\psi : H_k(K;\mathbb{Z}) \to H_k(M;\mathbb{Z})$.

(b) Integer analogues of the equivalences of Proposition 2.5.1 not involving (C) and (CM) hold. E.g. the integer analogue of (E)⇔(R) is Theorem 1.3.4; the integer analogue of (RH) is as follows: *there is a homomorphism $\psi : H_k(K;\mathbb{Z}) \to H_k(M;\mathbb{Z})$ such that $\omega_{\mathbb{Z}}(\psi) = v_{\mathbb{Z}}(K)$.*

(c) *The complex K is realizable by the matrix $H_{g,\mathbb{Z}}$ and some collection of vectors y_σ if and only if K is compatible to a matrix A such that* $\operatorname{rk} A \le 2g$. Cf. (CM)⇔(R) of Proposition 2.5.1. Let us deduce this statement from Lemmas 2.1.2 and 2.1.6.

Proof of the implication (⇒) is obtained from the proof of the implication (⇒) of Lemma 2.1.1 for I even by replacing 'even' with 'skew-symmetric' and '$\operatorname{rk} A \le \operatorname{rk} I$' with '$\operatorname{rk} A \le 2g$'.

Let us prove the implication (⇐). By Lemma 2.1.6 the number $r := \operatorname{rk} A$ is even and there is an $r\times n$ matrix Y such that $A = Y^T H_{r/2} Y$. Denote by Y_1 the $2g\times n$ matrix obtained from Y by adding $2g - r$ zeroes below each column of Y. Then $A = Y_1^T H_g Y_1$. Denote by (e_i) the basis in which I has the matrix $H_{g,\mathbb{Z}}$. Denote by y_σ the vector whose coordinates in the basis (e_i) form the corresponding column of matrix Y_1. Then (R_f) holds.

(c) A map $g : K \to M$ *induces* homomorphism $g_* : H_k(K;\mathbb{Z}) \to H_k(M;\mathbb{Z})$ by the formula $g_*[\alpha] := [g\alpha]$ for any $[\alpha] \in H_k(K;\mathbb{Z})$. It is simple and well-known that *any homomorphism $H_k(K;\mathbb{Z}) \to H_k(M;\mathbb{Z})$ is induced by some map $g : K \to M$.* Cf. Lemma 2.5.3.

(d) The analogue of Lemmas 2.5.5.abc hold for $\nu(g), v(K)$ and $\omega(g)$ replaced by $\nu_{\mathbb{Z}}(g), v_{\mathbb{Z}}(K)$ and $\omega_{\mathbb{Z}}(g)$, where $\omega_{\mathbb{Z}}(g)(\sigma,\tau) := (\overline{\sigma}\cup(-g'\sigma))\cdot(\overline{\tau}\cup(-g'\tau)) \in \mathbb{Z}$.

Remark 2.5.7. Consider the problems of existence of a $\mathbb{Z}_2$- or $\mathbb{Z}$-embedding $K \to M$ homotopic to given map $K \to M$.

(a) *Assume that $k \geq 3$ and M is simply connected. A map $K \to M$ is homotopic to an embedding if and only if the map is homotopic to a $\mathbb{Z}$-embedding.*

This is proved analogously to Theorem 1.3.1.a. The analogue for $k = 2$ is incorrect by Theorem 1.3.1.c. The analogue for $k = 1$ is correct analogously to the analogue for $k = 1$ of Theorem 1.3.1.a.

(b) (algorithms) Let T_M be a triangulation of M and $g : K \to T_M$ be a simplicial map whose restriction to the $(k-1)$-skeleton of K is null-homotopic. Then there are polynomial algorithms for checking the existence

• of a $\mathbb{Z}_2$-embedding $K \to M$ homotopic to g;

• of a $\mathbb{Z}$-embedding $K \to M$ homotopic to g, if M is oriented.

This follows from the criteria in (c,d). The algorithms are polynomial because the obtained system of equations is linear.

(c) Let $g : K \to M$ a general position PL map whose restriction to the $(k-1)$-skeleton of K is null-homotopic. Then following are equivalent.

(Eg) there is a $\mathbb{Z}_2$-embedding homotopic to g;

(EHg) $\nu(g)$ is cohomologous to the zero cocycle;

(RTg) for any $2k$-cycle $C \subset K^*$ and for some (equivalently, for any) maximal k-forest in K we have $\sum\limits_{\{\sigma,\tau\}\in C} g_*\widehat{\sigma} \cap_M g_*\widehat{\tau} = v_C(K)$.

(RIg) for any k-cycles $\alpha_1,\ldots,\alpha_q,\beta_1,\ldots,\beta_q$ in K such that $C := \sum\limits_{j=1}^{q}[\alpha_j \times \beta_j]$ is a $2k$-cycle in K^* we have $\sum\limits_{j=1}^{q} g_*\alpha_j \cap_M g_*\beta_j = v_C(K)$.

(RHg) $\omega(g_*) = v(K)$.

This (even with more equivalent conditions) is proved analogously to Proposition 2.5.1.

(d) Suppose that M is oriented. Let $g : K \to M$ a general position PL map whose restriction to the $(k-1)$-skeleton of K is null-homotopic. Then there is a $\mathbb{Z}$-embedding $K \to M$ homotopic to g if and only if $\nu_{\mathbb{Z}}(g)$ is cohomologous to the zero cocycle.

This is the particular case of Lemma 2.3.4 for $\nu_{\mathbb{Z}} = 0$. Other criteria (c) have also their integer analogues.

References

[Al] *A. Adrian Albert.* Symmetric and alternate matrices in an arbitrary field, I. Trans. Amer. Math. Soc., (1938) 43(3):386–436.

[AMS+] *S. Avvakumov, I. Mabillard, A. Skopenkov and U. Wagner.* Eliminating Higher-Multiplicity Intersections, III. Codimension 2, Israel J. Math. (2021). arxiv:1511.03501.

[Bi21] *A. I. Bikeev,* Criteria for integer and modulo 2 embeddability of graphs to surfaces, arXiv:2012.12070v2.

[Bo81] * *N. Bourbaki.* Elements of mathematics. Algebra.

[FH10] *M. Farber, E. Hanbury.* Topology of Configuration Space of Two Particles on a Graph, II. Algebr. Geom. Topol. 10 (2010) 2203–2227. arXiv:0903.2180.

[FK19] *R. Fulek, J. Kynčl,* $\mathbb{Z}_2$-genus of graphs and minimum rank of partial symmetric matrices, 35th Intern. Symp. on Comp. Geom. (SoCG 2019), Article No. 39; pp. 39:139:16, https://drops.dagstuhl.de/opus/volltexte/2019/10443/pdf/LIPIcs-SoCG-2019-39. We refer to numbering in arXiv version: arXiv:1903.08637.

[FKT] *M. H. Freedman, V. S. Krushkal and P. Teichner.* Van Kampen's embedding obstruction is incomplete for 2-complexes in $\mathbb{R}^4$, Math. Res. Letters. 1994. 1. P. 167-176.

[Ha69] *L. S. Harris,* Intersections and embeddings of polyhedra, Topology 8 (1969) 1–26.

[IF] * http://www.map.mpim-bonn.mpg.de/Intersection_form

[Ir65] *M. C. Irwin,* Embeddings of polyhedral manifolds, Ann. of Math. (2) 82 (1965) 1–14.

[Jo02] *C. M. Johnson.* An obstruction to embedding a simplicial n-complex into a $2n$-manifold, Topology Appl. 122:3 (2002) 581–591.

[Ko21] *E. Kogan.* On the rank of $\mathbb{Z}_2$-matrices with free entries on the diagonal, arXiv:2104.10668.

[Kr00] *V. S. Krushkal.* Embedding obstructions and 4-dimensional thickenings of 2-complexes, Proc. Amer. Math. Soc. 128:12 (2000) 3683–3691. arXiv:math/0004058.

[KS21] *E. Kogan and A. Skopenkov.* A short exposition of the Patak-Tancer theorem on non-embeddability of k-complexes in $2k$-manifolds, arXiv:2106.14010.

[La70] *L.L. Larmore*, The first obstruction to embedding a 1-complex in a 2-manifold, Illinois J. Math. 14 (1) (1970) 1-11.

[MC] https://en.wikipedia.org/wiki/Matrix_completion

[Me06] *S. A. Melikhov*, The van Kampen obstruction and its relatives, Proc. Steklov Inst. Math 266 (2009), 142-176 (= Trudy MIAN 266 (2009), 149-183), arXiv:math/0612082.

[MS17] *T. Maciazek, A. Sawicki.* Homology groups for particles on one-connected graphs J. Math. Phys. 58, 062103 (2017).

[MTW] *J. Matoušek, M. Tancer, U. Wagner.* Hardness of embedding simplicial complexes in $\mathbb{R}^d$, J. Eur. Math. Soc. 13:2 (2011), 259–295. arXiv:0807.0336.

[MW69] * *J. MacWilliams.* Orthogonal matrices over finite fields. Amer. Math. Monthly, (1969) 76:152–164.

[NKS] * *L. T. Nguyen, J. Kim, B. Shim.* Low-Rank Matrix Completion: A Contemporary Survey. arXiv:1907.11705.

[PT19] *P. Paták and M. Tancer.* Embeddings of k-complexes into $2k$-manifolds. arXiv:1904.02404.

[Sa91] *K. S. Sarkaria.* A one-dimensional Whitney trick and Kuratowski's graph planarity criterion, Israel J. Math. 73 (1991), 79–89. http://kssarkaria.org/docs/One-dimensional.pdf.

[Sc13] * *M. Schaefer.* Hanani-Tutte and related results. In Geometry — intuitive, discrete, and convex, Bolyai Soc. Math. Stud., 24 (2013), 259–299. http://ovid.cs.depaul.edu/documents/htsurvey.pdf

[Sh57] *A. Shapiro,* Obstructions to the embedding of a complex in a Euclidean space, I, The first obstruction, Ann. Math. 66 (1957), 256–269.

[Sk] * *A. Skopenkov.* Algebraic Topology From Algorithmic Viewpoint, draft of a book, mostly in Russian, http://www.mccme.ru/circles/oim/algor.pdf.

[Sk06] * *A. Skopenkov,* Embedding and knotting of manifolds in Euclidean spaces, London Math. Soc. Lect. Notes, 347 (2008) 248–342. arXiv:math/0604045.

[Sk14] * *A. Skopenkov,* Realizability of hypergraphs and Ramsey link theory, arXiv:1402.0658.

[Sk18] * *A. Skopenkov.* Invariants of graph drawings in the plane. Arnold Math. J., 6 (2020) 21–55; full version: arXiv:1805.10237.

[Sk20] * *A. Skopenkov,* Algebraic Topology From Geometric Viewpoint (in Russian), MCCME, Moscow, 2020 (2nd edition). Part of the book: http://www.mccme.ru/circles/oim/obstruct.pdf . Accepted for the English translation by Springer. Preprint of a part: https://www.mccme.ru/circles/oim/obstructeng.pdf.

[SS13] *M. Schaefer and D. Stefankovič.* Block additivity of $\mathbb{Z}_2$-embeddings. In Graph drawing, volume 8242 of Lecture Notes in Comput. Sci., 185–195. Springer, Cham, 2013. http://ovid.cs.depaul.edu/documents/genus.pdf

[vK32] *E. R. van Kampen*, Komplexe in euklidischen Räumen, Abh. Math. Sem. Hamburg, 9 (1933) 72–78; Berichtigung dazu, 152–153. English translation by Tu Tâm Nguễn-Phan: https://sites.google.com/site/tutamnguyenphan/van_Kampen.pdf

[Wa62] *C. T. C. Wall*, Classification of $(n-1)$-connected $2n$-manifolds, Ann. of Math., 75 (1962) 163–189.

[Wu58] *W. T. Wu.* On the realization of complexes in a euclidean space (in Chinese): I, Sci Sinica, 7 (1958) 251–297; II, Sci Sinica, 7 (1958) 365–387; III, Sci Sinica, 8 (1959) 133–150.

Books, surveys and expository papers in this list are marked by the stars.