1. Manifold Atlas : Regensburg Surgery Blockseminar 2012
Normal maps (Andreas Hermann and Michael Völkl)

Question. X a Poincaré complex. When does the Spivak normal fibration come from a vector bundle over X?

Definition 1.1. Let $\xi\colon E \to X$ be a vector bundle over a topological space X. We define the Thom space of ξ as

$$\mathrm{Th}(\xi) \coloneqq E \cup \{\infty\}$$

with the topology

$$\mathscr{O}_{\mathrm{Th}(\xi)} \coloneqq \mathscr{O}_E \cup \{\mathrm{Th}(\xi) \setminus A \mid A \subset E \text{closed}; \forall x \in X : A \cap E_x \text{compact}\}\,.$$

If a bundle metric on ξ is given, we define the disk bundle of ξ by

$$DE \coloneqq \{v \in E \mid \|v\| \leq 1\}$$

and the sphere bundle of ξ by

$$SE \coloneqq \{v \in E \mid \|v\| = 1\}.$$

Then $\mathrm{Th}(\xi)$ is homeomorphic to DE/SE, where the class of SE is the point ∞. If we view $S\xi\colon SE \to X$ as a spherical fibration then $\mathrm{Th}(\xi)$ is homeomorphic to the Thom space of $S\xi$ as defined in Talk 5.

Definition 1.2. For $i = 0, 1$ let $\xi_i\colon E_i \to X_i$ be vector bundles. A bundle map $(f, \overline{f})\colon \xi_0 \to \xi_1$ is a commutative diagram

$$\begin{array}{ccc} E_0 & \xrightarrow{\overline{f}} & E_1 \\ {\scriptstyle \xi_0}\downarrow & & \downarrow{\scriptstyle \xi_1} \\ X_0 & \xrightarrow{f} & X_1 \end{array}$$

such that $\overline{f}$ is a fiberwise linear isomorphism.

Every such bundle map $(f, \overline{f})$ induces canonically a map $\mathrm{Th}(\overline{f})\colon \mathrm{Th}(\xi_0) \to \mathrm{Th}(\xi_1)$.

If M is a closed manifold of dimension n and $i\colon M \to \mathbb{R}^{n+k}$ an embedding, we denote the normal bundle of M corresponding to i by $\nu(M, i)$. It is the subbundle of $i^*T\mathbb{R}^{n+k}$ given by the orthogonal complement of the tangent bundle with respect to the Euclidean metric on $i^*T\mathbb{R}^{n+k}$.

Definition 1.3. Let $\xi\colon E \to X$ be a vector bundle of rank k over a CW complex X. We define the ξ-bordism set as the set of equivalence classes

$$\Omega_n(\xi) \coloneqq \left\{ [M, i, f, \overline{f}] \;\middle|\; \begin{array}{l} M \text{ closed manifold of dimension } n, \\ i\colon M \to \mathbb{R}^{n+k} \text{ embedding}, \\ (f, \overline{f})\colon \nu(M, i) \to \xi \text{ bundle map} \end{array} \right\},$$

where we identify $(M_0, i_0, f_0, \overline{f}_0) \sim (M_1, i_1, f_1, \overline{f}_1)$ iff

(1) there exists a compact manifold W of dimension $n + 1$ such that $\partial W = \partial_0 W \amalg \partial_1 W$

(2) there exists an embedding $I\colon W \to \mathbb{R}^{n+k} \times [0, 1]$ such that $I^{-1}(\mathbb{R}^{n+k} \times \{j\}) = \partial_j W$ and W meets $\mathbb{R}^{n+k} \times \{j\}$ transversally for $j = 0, 1$

(3) there exists a bundle map

$$\begin{array}{ccc} \nu(W, I) & \xrightarrow{\overline{F}} & E \times [0, 1] \\ \downarrow & & \downarrow{\scriptstyle \xi \times \mathrm{id}} \\ W & \xrightarrow{F} & X \times [0, 1] \end{array}$$

such that $F(\partial_j W) \subset X \times \{j\}$ for $j = 0, 1$

(4) for $j = 0, 1$ there exist diffeomorphisms $U_j \colon \mathbb{R}^{n+k} \to \mathbb{R}^{n+k} \times \{j\}$ such that
  (a) $U_j|_{M_j}$ is a diffeomorphism $M_j \to \partial_j W$
  (b) $F \circ U_j|_{M_j} = f_j$
  (c) the induced bundle map

$$\begin{array}{ccc} \nu(M_j, i_j) & \xrightarrow{\nu(U_j)} & \nu(W, I)|_{\partial_j W} \\ \downarrow & & \downarrow \\ M_j & \xrightarrow{U_j} & \partial_j W \end{array}$$

satisfies $\overline{F} \circ \nu(U_j) = \overline{f}_j$.

Let $i \colon M \to \mathbb{R}^{n+k}$ be as above and let $N(M)$ be a closed tubular neighbourhood of M in $\mathbb{R}^{n+k}$. Then there exists a diffeomorphism

$$u \colon N(M) \to D\nu(M, i).$$

We define the collapse map

$$c \colon S^{n+k} \to \mathbb{R}^{n+k} \cup \{\infty\} \to \mathrm{Th}(\nu(M, i))$$

as the map, which is equal to u on the interior of $N(M)$ and sends everything else to the point ∞.

Theorem 1.4 (Pontrjagin-Thom construction)**.** *Let $\xi \colon E \to X$ be a vector bundle of rank k over a CW complex X. Then*

$$P_n(\xi) \colon \Omega_n(\xi) \to \pi_{n+k}(\mathrm{Th}(\xi))$$

$$[M, i, f, \overline{f}] \mapsto [S^{n+k} \xrightarrow{c} \mathrm{Th}(\nu(M, i)) \xrightarrow{\mathrm{Th}(\overline{f})} \mathrm{Th}(\xi)]$$

is a bijection and natural in ξ.

Proof. We consider the special case that X is a compact manifold and we want to find $Q \colon \pi_{n+k}(\mathrm{Th}(\xi)) \to \Omega_n(\xi)$, which is an inverse to $P_n(\xi)$. The idea is as follows. If $[g] \in \pi_{n+k}(\mathrm{Th}(\xi))$, then we can find a representative $g \colon S^{n+k} \to \mathrm{Th}(\xi)$ with the following properties:

(1) g is transverse to the zero section $X \subset \mathrm{Th}(\xi)$, in particular $M := g^{-1}(X)$ is a closed manifold of dimension n and the inclusion $i \colon M \to \mathbb{R}^{n+k} \subset S^{n+k}$ is an embedding
(2) an open tubular neighbourhood $N(M) \subset \mathbb{R}^{n+k}$ of M can be chosen such that $N(M) = g^{-1}(E)$
(3) there exists a diffeomorphism $h \colon \nu(M, i) \to N(M)$ such that the composition

$$\nu(M, i) \xrightarrow{h} N(M) \xrightarrow{g} E$$

is a fiberwise linear isomorphism.

We define $Q([g]) := [M, i, g|_M, g \circ h]$. Then $Q = P_n(\xi)^{-1}$. □

Definition 1.5. Let X be a CW complex. We define the X-bordism set as the set of equivalence classes

$$\Omega_n(X) := \{[M, f] \mid M \text{closed oriented manifold of dimension } n, f \colon M \to X\},$$

where we identify $(M_0, f_0) \sim (M_1, f_1)$ iff

(1) there exists a compact oriented manifold W of dimension $n + 1$ and there exists an orientation preserving diffeomorphism $\partial W \cong M_0 \amalg M_1^-$, where M_1^- denotes M_1 with the reversed orientation.

(2) there exists $F\colon W \to X$ such that $F|_{\partial W}$ is given by (f_0, f_1) under the diffeomorphism from above.

With the addition $[M_0, f_0] + [M_1, f_1] := [M_0 \amalg M_1, (f_0, f_1)]$ the set $\Omega_n(X)$ is an abelian group.

Let $E_k \xrightarrow{\xi_k} \mathrm{BSO}(k)$ be the universal oriented vector bundle of rank k. We define the vector bundle $E_k \times X \xrightarrow{\gamma_k} \mathrm{BSO}(k) \times X$ by $\gamma_k := \xi_k \times \mathrm{id}_X$ and we define the maps

$$V_k\colon \Omega_n(\gamma_k) \to \Omega_n(X), \quad [M, i, f, \overline{f}] \mapsto [M, \mathrm{pr}_X \circ f].$$

Let $\underline{\mathbb{R}} \to \mathrm{BSO}(k)$ be the trivial vector bundle of rank 1. By the classifying property of $\mathrm{BSO}(k+1)$ we obtain bundle maps $(j_k, \overline{j}_k)\colon \xi_k \oplus \underline{\mathbb{R}} \to \xi_{k+1}$ which are unique up to homotopy. These bundle maps induce stabilization maps $\Omega_n(\gamma_k) \to \Omega_n(\gamma_{k+1})$ which are compatible with the maps V_k (see Exercise on Thom spaces). Therefore we obtain a map

$$V\colon \operatorname{colim}_{k\to\infty} \Omega_n(\gamma_k) \to \Omega_n(X),$$

which is a bijection. We also obtain a bijection (see Exercise)

$$\operatorname{colim}_{k\to\infty} \Omega_n(\gamma_k) \to \operatorname{colim}_{k\to\infty} \pi_{n+k} \mathrm{Th}(\gamma_k).$$

Together this implies the following.

Theorem 1.6 (Pontrjagin-Thom construction and oriented bordism)**.** *There is an isomorphism of abelian groups*

$$P\colon \Omega_n(X) \to \operatorname{colim}_{k\to\infty} \pi_{n+k} \mathrm{Th}(\gamma_k)$$

which is natural in X.

We briefly recall some constructions. Define

$$G(k) := \left\{ S^{k-1} \to S^{k-1} \text{ homotopy equivalence} \right\} .$$

This is a topological monoid (with composition and compact-open topology). There is a suspension map $G(k) \xrightarrow{\Sigma} G(k+1); f \mapsto \Sigma f$. Define

$$G := \operatorname{colim} G(k) .$$

This is again a topological monoid. We do the same for $O(k)$. There is a suspension map

$$\Sigma\colon O(k) \longrightarrow O(k+1),$$
$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

and we define $O := \operatorname{colim} O(k)$. There is a map

$$J_k\colon O(k) \longrightarrow G(k),$$
$$A \mapsto A_{|S^{k-1}}$$

which induces $J\colon O \to G$. We apply the functor B and get $J\colon \mathrm{BO} \to BG$.

Definition 1.7 (Normal k-invariant)**.** X a Poincaré complex, $k \in \mathbb{N}_0$. A normal k-invariant (ξ, c) is a vector bundle $\xi : E \to X$, $c \in \pi_{n+k} \mathrm{Th}(\xi)$ such that $[X] = H_n(\xi)(U_\xi \cap h(c))$. There is a notion of equivalence/bordism:

$(\xi_1, c_1) \sim (\xi_2, c_2) \Leftrightarrow \exists$ vector bundle iso

$$\begin{array}{ccc} \xi_1 & \longrightarrow & \xi_2 \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

such that $\pi_{n+k}(\mathrm{Th}(\overline{f}))[c_1] = [c_2]$. We denote the set of equivalence classes by $\mathscr{T}_n(X, k)$. There is a suspension map

$$\Sigma\colon \mathscr{T}_n(X,k) \longrightarrow \mathscr{T}_n(X,k+1),$$
$$(\xi, c) \mapsto (\xi \oplus \underline{\mathbb{R}}, \overline{c})$$

(use $\pi_{n+k}(T(\xi \oplus \underline{\mathbb{R}}) \cong \pi_{n+k}(\Sigma\,\mathrm{Th}(\xi)))$.

The elements in $\mathscr{T}_n(X) \coloneqq \operatorname{colim}_{k\to\infty} \mathscr{T}_n(X,k)$ are called the normal invariants. Recall

- BG classifies stable spherical fibration up to homotopy
- X a Poincaré complex $\Rightarrow$ SNF $E \to X \Rightarrow s_X : X \to BG$

Theorem 1.8 (3.43)**.** *X a Poincaré complex. Then there is a lift $\tilde{s}_X$ of s_X along $J : BO \to BG$ if and only if $\mathscr{T}_n(X) \neq \emptyset$.*

Define $\mathrm{G/O} \coloneqq \mathrm{hofiber}(J\colon \mathrm{BO} \to \mathrm{BG})$

Theorem 1.9 (3.45)**.** *X a Poincaré complex, $\mathscr{T}_n(X) \neq \emptyset$. Then there is a group structure on $[X, \mathrm{G/O}]$ and there is a free and transitive action of $[X, \mathrm{G/O}]$ on the set of lifts in the theorem above. Furthermore there is a bijection of $\mathscr{T}_n(X)$ and the set of lifts.*

Proof. The first part is an exercise. The second can be found in Lück. □

Remark 1.10. Note that this induces a bijection between $\mathscr{T}_n(X)$ and $[X, \mathrm{G/O}]$ after fixing a basepoint. In the case that X is a closed connected manifold there is a canonical basepoint (induced by the stable normal bundle).

Now we replace the homotopy class c in the definition of a normal invariant by geometric bordism data:

Definition 1.11 (Normal k-map)**.** Let X be a Poincaré complex $k \in \mathbb{N}_0$ fixed. A normal k-map $(\xi, M, i, f, \overline{f})$ consists of

- a closed n-dimensional manifold M
- a vector bundle $\xi\colon E \to X$ of rank k
- an embedding $i\colon M \to \mathbb{R}^{n+k}$
- a bundle morphism

$$\begin{array}{ccc} \nu(M) & \xrightarrow{\overline{f}} & E \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

  such that f has degree one.

There is a notion of equivalence/bordism (see exercises). The set of equivalence classes is $\mathscr{N}_n(X, k)$. Again we have a suspension map

$$\Sigma : \mathscr{N}_n(X,k) \to \mathscr{N}_n(X,k+1); (\xi, M, i, f, \overline{f}) \mapsto (\xi \oplus \underline{\mathbb{R}}, M, i', f, \overline{f}')$$

where $i'\colon M \xrightarrow{i} \underline{\mathbb{R}}^{n+k} \to \underline{\mathbb{R}}^{n+k} \oplus \underline{\mathbb{R}}$ and $\overline{f}'\colon \nu(M,i') = \nu(M,i) \oplus \underline{\mathbb{R}} \xrightarrow{\overline{f}\oplus \mathrm{id}} \xi \oplus \underline{\mathbb{R}}$. Hence we can define: $\mathscr{N}_n(X) \eqqcolon \operatorname{colim} \mathscr{N}_n(X,k)$, the set of normal maps.

Theorem 1.12 (3.48)**.** *There are bijections*

$$p_k\colon \mathscr{N}_n(X,k) \xrightarrow{\simeq} \mathscr{T}_n(X,k)\ ,$$
$$p\colon \mathscr{N}_n(X) \xrightarrow{\simeq} \mathscr{T}_n(X)\ .$$

Proof. Pontrjagin-Thom construction □

Now we can work with geometric data, but they are extrinsic since they use the stable normal bundle. But we can work intrinsically by using the stable tangent bundle:

Definition 1.13. (normal map w.r.t. the tangent bundle)
Let X be a PC.
A normal map w.r.t. the tangent bundle $(\xi, M, a, f, \bar{f})$ (of degree one) consists of

- a vector bundle $\xi : E \to X$
- a closed n-dim manifold M
- a bundle map

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}}^a & \xrightarrow{\bar{f}} & E \\ \downarrow & & \downarrow \xi \\ M & \xrightarrow{\text{f}} & X \end{array}$$

such that f has degree one.

We define $(\xi_0, M_0, a_0, f_0, \bar{f}_0) \sim (\xi_1, M_1, a_1, f_1, \bar{f}_1)$ iff

(1) There exists a compact $(n+1)$-dimensional manifold W whose boundary can be written as $\partial W = \partial_0 W \sqcup \partial_1 W$.
(2) There exists a vector bundle η: $E' \to X \times [0,1]$ and there exists a bundle map

$$\begin{array}{ccc} TW \oplus \underline{\mathbb{R}}^b & \xrightarrow{\bar{F}} & E' \\ \downarrow & & \downarrow \eta \\ W & \xrightarrow{\text{F}} & X \times [0,1] \end{array}$$

such that $F(\partial_j W) \subset X \times \{j\}$ and

$$F: \quad (W, \partial W) \to (X \times [0,1], X \times \{0,1\})$$

has degree one.
(3) There exist diffeomorphisms $M_j \overset{U_j}{\to} \partial_j W$ such that $F \circ U_j = f_j$ for $j = 0, 1$.
(4) There exist bundle isomorphisms $\xi_j \oplus \underline{\mathbb{R}}^{b-a_j+1} \overset{v_j}{\to} \eta|_{X \times \{j\}}$ such that the following diagram commutes

$$\begin{array}{ccc} TM_j \oplus \underline{\mathbb{R}} \oplus \underline{\mathbb{R}}^b & \xrightarrow{\bar{f}_j \oplus \mathrm{id}_{\underline{\mathbb{R}}^{b-a_j+1}}} & \xi_j \oplus \underline{\mathbb{R}}^{b-a_j+1} \\ \downarrow {\scriptstyle TU_j \oplus n_j \oplus \mathrm{id}_{\underline{\mathbb{R}}^b}} & & \downarrow {\scriptstyle v_j} \\ TW|_{\partial_j W} \oplus \underline{\mathbb{R}}^a & \xrightarrow{\bar{F}|_{\partial_j W}} & \eta|_{X \times \{j\}} \end{array}$$

Here TU_j: $TM_j \to TW|_{\partial_j W}$ is the differential of U_j and n_j: $\underline{\mathbb{R}} \to TW|_{\partial_j W}$ is given by an inward normal field of $TW|_{\partial_j W}$.

Definition 1.14. The set of equivalence classes of normal maps w.r.t. the tangent bundle is denoted as $\mathcal{N}_n^T(X)$.

Lemma 1.15. *(Lemma 3.51)*
Let X be a PC. Then there is a bijection $\mathcal{N}_n^T(X) \cong \mathcal{N}_n(X)$.

Proof. We construct a map $\Phi_n(k) : \mathcal{N}_n(X, k) \to \mathcal{N}_n^T(X)$.

- So fix a normal k-map $(\xi, M, i, f, \bar{f})$.

- Since X is compact, there is a vector bundle η and an isomorphism $\eta\oplus\xi \overset{u}{\to} \underline{\mathbb{R}}^a$ for some $a \in \mathbb{N}_0$.
- We have an explicit isomorphism $\nu(M,i) \oplus TM \overset{v}{\to} \underline{\mathbb{R}}^{n+k}$.
- Hence we have:

$$f^*\eta \oplus \underline{\mathbb{R}}^{n+k} \cong f^*\eta \oplus \nu(M,i) \oplus TM \cong f^*\eta \oplus f^*\xi \oplus TM \cong \underline{\mathbb{R}}^a \oplus TM$$

- This isomorphism (from right to left) is a map $TM \oplus \underline{\mathbb{R}}^a \to f^*(\eta \oplus \underline{\mathbb{R}}^{n+k})$ covering $id : M \to M$.
- So it yields an isomorphism $\bar{g} : TM \oplus \underline{\mathbb{R}}^a \to \eta \oplus \underline{\mathbb{R}}^{n+k}$ covering $f : M \to X$.
- We define: $\Phi_n(k)(\xi, M, i, f, \bar{f}) := (\eta\oplus\underline{\mathbb{R}}^{n+k}, M, a, f, \bar{g})$ (which is compatible with the equivalence relations).

This definitions of the $\Phi_n(k)$ fit together to give a map $\Phi_n : \mathscr{N}_n(X) \to \mathscr{N}_n^T(X)$. The inverse map is constructed analogously. □

Now we put everything together and formulate it for a manifold:

Theorem 1.16 (3.52). *M a closed connected n-dimensional manifold*

$$[M, \mathrm{G/O}] \cong \mathscr{T}_n(M) \cong \mathscr{N}_n(M) \cong \mathscr{N}_n^T(M)$$