Manifold Atlas: supplementary material for the page **Formal group laws and genera**

Proposition. *The following relation holds in $U^2(X)$:*

$$u +_H v = F_U(u, v) = u + v + \sum_{k\geq 1,\, l\geq 1} \alpha_{kl}\, u^k v^l, \tag{1}$$

where the coefficients $\alpha_{kl} \in \Omega_U^{-2(k+l-1)}$ do not depend on X. The series $F_U(u, v)$ given by (1) is a formal group law over the ring $\Omega_U = \Omega_U^*$.

Proof. We first do calculations with the universal example $X = \mathbb{C}P^\infty \times \mathbb{C}P^\infty$. Then

$$U^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = \Omega_U^*[[\underline{u}, \underline{v}]],$$

where $\underline{u}, \underline{v}$ are canonical geometric cobordisms given by the projections of $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$ onto its factors. We therefore have the following relation in $U^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$:

$$\underline{u} +_H \underline{v} = \sum_{k,l\geq 0} \alpha_{kl}\, \underline{u}^k \underline{v}^l, \tag{2}$$

where $\alpha_{kl} \in \Omega_U^{-2(k+l-1)}$.

Now let the geometric cobordisms $u, v \in U^2(X)$ be given by maps $f_u, f_v \colon X \to \mathbb{C}P^\infty$ respectively. Then $u = (f_u \times f_v)^*(\underline{u})$, $v = (f_u \times f_v)^*(\underline{v})$ and $u +_H v = (f_u \times f_v)^*(\underline{u} +_H \underline{v})$, where $f_u \times f_v \colon X \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$. Applying the Ω_U^*-module map $(f_u \times f_v)^*$ to (2) we obtain the required formula (1). The fact that $F_U(u, v)$ is a formal group law follows directly from the properties of the group multiplication $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$. □

Taras Panov
Department of Geometry and Topology
Faculty of Mathematics and Mechanics
Moscow State University, Leninskie Gory
119991 Moscow RUSSIA

E-mail address: tpanov@higeom.math.msu.su