1 The signature of the Milnor hypersurfaces

Let H_{ij} be the Milnor hypersurface of complex codimension 1 in $\mathbb{C}P^i \times \mathbb{C}P^j$. We are interested in computing $\text{sign}\,(H_{ij})$. The real dimension of H_{ij} is $2(i+j-1)$, so unless $i+j$ is odd, $\text{sign}\,(H_{ij}) = 0$.

Theorem 1. *Assume* $0 \leq i \leq j$.

$$sign\,(H_{ij}) = \begin{cases} 1 & : i \text{ even, } j \text{ odd} \\ 0 & : \text{otherwise} \end{cases}$$

Remark. $H_{ij} \cong H_{ji}$, so this is a computation for all Milnor hypersurfaces. We suspect that there should be some geometric reason why this formula is so pleasant, but we haven't found one yet.

Proof. Let $F_U(u,v)$ be the formal group law of geometric cobordisms. Panov 4.1 and Hirzebruch tell us that the exponential of $\text{sign}\,(F_U(u,v))$ is the formal power series for $\tanh(x)$, so the logarithm is $\tanh^{-1}(x)$:

$$\tanh^{-1}(\text{sign}\,(F_U(u,v))) = \tanh^{-1}(u) + \tanh^{-1}(v). \tag{1}$$

Applying tanh to both sides and using the sum formula for tanh,

$$\begin{aligned} \text{sign}\,(F(u,v)) &= \tanh(\tanh^{-1}(u) + \tanh^{-1}(v)) & (2) \\ &= \frac{u+v}{1+uv} & (3) \\ &= (u+v)\sum_{k=0}^{\infty}(-uv)^k & (4) \\ &= \sum_{k=0}^{\infty}(-1)^k u^{k+1} v^k + \sum_{k=0}^{\infty}(-1)^k u^k v^{k+1}. & (5) \end{aligned}$$

We display (5) for future reference:

	0	1	2	3	4	5	6	7	8
0	0	1	0	0	0	0	0	0	0
1	1	0	-1	0	0	0	0	0	0
2	0	-1	0	1	0	0	0	0	0
3	0	0	1	0	-1	0	0	0	0
4	0	0	0	-1	0	1	0	0	0
5	0	0	0	0	1	0	-1	0	0
6	0	0	0	0	0	-1	0	1	0
7	0	0	0	0	0	0	1	0	-1
8	0	0	0	0	0	0	0	-1	0

Now we use Buchstaber's geometric description of $F_U(u,v)$ (Panov 3.2),

$$F_U(u,v) = \frac{\sum_{i,j\geq 0}[H_{ij}]u^i v^j}{\left(\sum_{r\geq 0}[\mathbb{C}P^r]u^r\right)\left(\sum_{s\geq 0}[\mathbb{C}P^s]v^s\right)} \tag{6}$$

Applying the signature operator to this equation, and using $\operatorname{sign}\left(\mathbb{C}P^{2r-1}\right) = 0$ and $\operatorname{sign}\left(\mathbb{C}P^{2r}\right) = 1$,

$$\begin{aligned}
\operatorname{sign}(F_U(u,v)) &= \frac{\sum_{i,j\geq 0}\operatorname{sign}(H_{ij})\, u^i v^j}{\left(\sum_{r\geq 0} u^{2r}\right)\left(\sum_{s\geq 0} v^{2s}\right)} && (7)\\
&= \frac{\sum_{i,j\geq 0}\operatorname{sign}(H_{ij})\, u^i v^j}{\left(\frac{1}{1-u^2}\right)\left(\frac{1}{1-v^2}\right)} && (8)\\
&= (1-u^2)(1-v^2)\sum_{i,j\geq 0}\operatorname{sign}(H_{ij})\, u^i v^j && (9)\\
&= \sum_{i,j\geq 0}(\operatorname{sign}(H_{ij}) - \operatorname{sign}\left(H_{(i-2)j}\right) - \operatorname{sign}\left(H_{i(j-2)}\right) + \operatorname{sign}\left(H_{(i-2)(j-2)}\right))u^i v^j && (10)
\end{aligned}$$

Here we interpret $\operatorname{sign}(H_{ij}) = 0$ if i or j is negative. Setting (5) and (10) equal,

$$\operatorname{sign}(H_{ij}) = \operatorname{sign}\left(H_{(i-2)j}\right) + \operatorname{sign}\left(H_{i(j-2)}\right) - \operatorname{sign}\left(H_{(i-2)(j-2)}\right) + \gamma_{ij} \tag{11}$$

where γ_{ij} is the value in the table for (5).
(11) gives us a recursive formula for $\operatorname{sign}(H_{ij})$, so now we have to find a closed form.
Using that $\operatorname{sign}(H_{ij}) = 0$ if i or j is negative, together with $\gamma_{01} = 1$, we fill in the first row recursively and use symmetry to fill in the first column:

	0	1	2	3	4	5	6	7	8
0	0	1	0	1	0	1	0	1	0
1	1								
2	0								
3	1								
4	0								
5	1								
6	0								
7	1								
8	0								

Along the second row, $\operatorname{sign}(H_{11}) = 0, \operatorname{sign}(H_{21}) = 1 + 0 - 0 + (-1) = 0$, and then the vanishing of the (-1)st row lets us conclude that

	0	1	2	3	4	5	6	7	8
0	0	1	0	1	0	1	0	1	0
1	1	0	0	0	0	0	0	0	0
2	0	0							
3	1	0							
4	0	0							
5	1	0							
6	0	0							
7	1	0							
8	0	0							

$\text{sign}\,(H_{22}) = 0, \text{sign}\,(H_{32}) = 0 + 1 - 1 + 1 = 1$, after which we compute either $0 + 0 - 0 = 0$ or $1 + 1 - 1 = 1$:

	0	1	2	3	4	5	6	7	8
0	0	1	0	1	0	1	0	1	0
1	1	0	0	0	0	0	0	0	0
2	0	0	0	1	0	1	0	1	0
3	1	0	1						
4	0	0	0						
5	1	0	1						
6	0	0	0						
7	1	0	1						
8	0	0	0						

From here we note that since all values along the (-1)st row and column vanish, $\text{sign}\,(H_{ij}) = \text{sign}\,\left(H_{(i-2)(j-2)}\right)$ along the first three rows and columns. Since $\gamma_{ij} = \gamma_{(i-2)(j-2)}$ as well, our recursive formula lets us conclude that $\text{sign}\,(H_{ij}) = \text{sign}\,\left(H_{(i-2)(j-2)}\right)$ for $i, j \geq 2$:

	0	1	2	3	4	5	6	7	8
0	0	1	0	1	0	1	0	1	0
1	1	0	0	0	0	0	0	0	0
2	0	0	0	1	0	1	0	1	0
3	1	0	1	0	0	0	0	0	0
4	0	0	0	0	0	1	0	1	0
5	1	0	1	0	1	0	0	0	0
6	0	0	0	0	0	0	0	1	0
7	1	0	1	0	1	0	1	0	0
8	0	0	0	0	0	0	0	0	0

Using that $\text{sign}\,(H_{ij}) = \text{sign}\,\left(H_{(i-2)(j-2)}\right)$, along with the values for the first two rows, gives the form stated in the theorem.

□

Let's verify this for small i, j using the Hirzebruch signature formula. When $j = 0$, $H_{i0} \cong \mathbb{C}P^{i-1}$, so our formula is correct along the 0th row and column. The first interesting cases are $\text{sign}\,(H_{21}) = 0, \text{sign}\,(H_{32}) = 1$.
Following Stong, we note that since the fundamental class for $H_{ij} \hookrightarrow \mathbf{C}P^i \times \mathbf{C}P^j$ is Poincaré dual to the cohomology class $x + y$, and setting $\overline{x} = i^*(x), \overline{y} = i^*(y)$, the total Chern class of H_{ij} is given by

$$c(H_{ij}) = \frac{i^*(c(\mathbf{C}P^i \times \mathbf{C}P^j))}{1 + \overline{x} + \overline{y}} = \frac{(1+\overline{x})^{i+1}(1+\overline{y})^{j+1}}{1 + \overline{x} + \overline{y}}$$

and therefore the total Pontryagin class is

$$p(H_{ij}) = \frac{(1+\overline{x}^2)^{i+1}(1+\overline{y}^2)^{j+1}}{1 + (\overline{x} + \overline{y})^2}.$$

Hirzebruch tells us that

$$\begin{aligned}
\text{sign}\,(H_{ij}) &= \langle L(p(H_{ij})), [H_{ij}] \rangle \\
&= \left\langle \left(\frac{\overline{x}}{\tanh \overline{x}}\right)^{i+1} \left(\frac{\overline{y}}{\tanh \overline{y}}\right)^{j+1} \frac{\tanh\,(\overline{x}+\overline{y})}{\overline{x}+\overline{y}}, [H_{ij}] \right\rangle \\
&= \left\langle i^* \left(\left(\frac{x}{\tanh x}\right)^{i+1} \left(\frac{y}{\tanh y}\right)^{j+1} \frac{\tanh\,(x+y)}{x+y} \right), [H_{ij}] \right\rangle \\
&= \left\langle \left(\frac{x}{\tanh x}\right)^{i+1} \left(\frac{y}{\tanh y}\right)^{j+1} \frac{\tanh\,(x+y)}{x+y}(x+y), [\mathbf{C}P^i \times \mathbf{C}P^j] \right\rangle,
\end{aligned}$$

where the last equality is the naturality of Poincaré duality. But this is the coefficient of $x^i y^j$ in $\left(\frac{x}{\tanh x}\right)^{i+1} \left(\frac{y}{\tanh y}\right)^{j+1} \tanh\,(x+y)$.
Now we use the power series expansions

$$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} + ... \tag{12}$$

$$\frac{x}{\tanh x} = 1 + \frac{x^2}{3} + ... \tag{13}$$

When $i = 1, j = 2$, we want the coefficient of xy^2 in

$$\begin{aligned}
\left(\frac{x}{\tanh x}\right)^2 \left(\frac{y}{\tanh y}\right)^3 \tanh\,(x+y) &= (1)^2(1 + \frac{y^2}{3})^3((x+y) - \frac{(x+y)^3}{3}) \\
&= (1+y^2)(x+y-xy^2)
\end{aligned}$$

whose xy^2 coefficient is $-1 + 1 = 0$, so we recover $\text{sign}\,(H_{12}) = 0$.

When $i = 2, j = 3$, we want the coefficient of x^2y^3 in

$$\begin{aligned}\left(\frac{x}{\tanh x}\right)^3\left(\frac{y}{\tanh y}\right)^4 \tanh(x+y) &= (1+\frac{x^2}{3})^3(1+\frac{y^2}{3})^4((x+y) - \frac{(x+y)^3}{3} + \frac{2(x+y)^5}{15}) \\ &= (1+x^2)(1+\frac{4y^2}{3})(x+y-x^2y-xy^2-\frac{y^3}{3}+\frac{2\cdot 10\cdot x^2y^3}{15}) \\ &= (1+x^2+\frac{4y^2}{3}+\frac{4x^2y^2}{3})(x+y-x^2y-xy^2-\frac{y^3}{3}+\frac{4x^2y^3}{3})\end{aligned}$$

whose x^2y^3 coefficient is $\frac{4}{3} - \frac{1}{3} - \frac{4}{3} + \frac{4}{3} = 1$, and we recover $\operatorname{sign}(H_{23}) = 1$.