1. MANIFOLD ATLAS : REGENSBURG SURGERY BLOCKSEMINAR 2012

MANIFOLDS WITH BOUNDARY AND SIMPLE SURGERY OBSTRUCTIONS (BERND AMMANN)

In the previous talks a map $f : M \to X$ from a manifold to a Poincaré complex was given, equipped with some normal data, and we wanted to modify it to a homotopy equivalence.

In particular we had the following theorem:

Theorem 1.1 (Surgery Obstruction theorem)**.** *Let $(\overline{f}, f)$ be a normal map with underlying map $f\colon M \to X$.*

1. *We obtain a surgery obstruction $\sigma(\overline{f}, f) \in L_n(\mathbb{Z}\pi, w)$ which only depends on the normal bordism class of $(\overline{f}, f)$ with normal data.*
2. *Suppose $n \geq 5$. $\sigma(\overline{f}, f) = 0$ in $L_n(\mathbb{Z}\pi, w)$ iff $(\overline{f}, f)\colon (TM \oplus \mathbb{R}^a, M) \to (\xi, X)$ can be simplified by a finite number of surgeries to $(\overline{f}', f')\colon (TM' \oplus \mathbb{R}^a, M') \to (\xi, X)$ where $f'\colon M' \to X$ is a homotopy equivalence.*

1.1. **Variations of previous results.** New question $h\colon N_0 \to N_1$ shall be given. When is N_0 diffeo to N_1?

Approach:

1. Construct $(f, h \amalg \mathrm{id})\colon (M, N_0 \amalg N_1) \to (N_1 \times [0,1] = X, N_1 \amalg N_1)$
2. Simply $(f, h \amalg \mathrm{id})$ and normal data to a homotopy equivalence. Then M is an h-cobordism.

We need bordisms between manifolds with boundary.

This approach then has to be modified in such a way that we obtain a trivial bordism, and not just an h-cobordism.

Definition 1.2 (Manifold triad)**.** A manifold triad $(W, \partial_0 W, \partial_1 W)$ is a manifold with boundary $\partial W = \partial_0 W \cup \partial_1 W$ where $\partial_0 W, \partial_1 W$ submanifolds with boundary ∂W

$$\partial\partial_1 W = \partial\partial_0 W = \partial_0 W \cap \partial_1 W$$

Definition 1.3 (Poincaré triad)**.** A m-dimensional Poincaré triad $(X, \partial_0 X, \partial_1 X)$ is finite m-dimensional Poincaré pair $(X, \partial X)$ such that $\partial X = \partial_0 X \cup_{\partial_0 X \cap \partial_1 X} \partial_1 X$ such that $\partial_0 X \cap \partial_1 X$ is of dimension $(m-2)$. $(\partial_0 X, \partial_0 \cap \partial_1 X)$ and $(\partial_1 X, \partial_0 \cap \partial_1 X)$ Poincaré pairs of dimension $(m-1)$.

Definition 1.4 (Normal map)**.** Normal map for manifolds with boundary

$$\begin{array}{ccc} TM \oplus \mathbb{R}^a & \xrightarrow[\text{fw. iso}]{\overline{f}} & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow[\text{degree } 1]{f} & X \\ \uparrow & & \uparrow \\ \partial M & \xrightarrow[\text{htpq eq}]{\partial f} & \partial X \end{array}$$

Definition 1.5. A normal null bordism of such a normal map consists of

(1)

$$(W, \partial_0 W, \partial_1 W) \xrightarrow{(F, \partial_0 F, \partial_1 F)} (Y, \partial_0 Y, \partial_1 Y)$$

Here $(W, \partial_0 W, \partial_1 W)$ is an $(m+1)$-dimensional manifold triad and $(Y, \partial_0 Y, \partial_1 Y)$ is an $(m+1)$-dimensional Poincaré triad. Furthermore we require $\deg F = 1$ and that $\partial_1 F$ is a homotopy equivalence.

(2) $(u, \partial u)\colon (M, \partial M) \to (\partial_0 W, \partial\partial_0 W)$ orientation preserving diffeomorphism.
(3) $(v, \partial v)\colon (X, \partial X) \to (\partial_0 Y, \partial_0 Y \cap \partial_1 Y)$ homeomorphism, and isomorphism of Poincaré pairs. $\partial_0 F \circ u = v \circ f$.
(4) bundle and compatibility data

Theorem 1.6 (Surgery Obstruction theorem for manifolds with boundary)**.** *Let $(\overline{f}, f)$ be a normal map with underlying map $(f, \partial f)\colon (M, \partial M) \to (X, \partial X)$ such that ∂f is a homotopy equivalence.*

(1) *We obtain a surgery obstruction $\sigma(\overline{f}, f) \in L_n(\mathbb{Z}\pi, w)$ which only depends on the normal bordism class of $(\overline{f}, f)$*
(2) *Suppose $n \geq 5$. $\sigma(\overline{f}, f) = 0$ in $L_n(\mathbb{Z}\pi, w)$ iff $(\overline{f}, f, \partial f)\colon (TM \oplus \underline{\mathbb{R}}^{a}, M, \partial M) \to (\xi, X, \partial X)$ can be simplified by a finite number of surgeries to a normal map $(\overline{f}', f', \partial f')\colon (TM' \oplus \underline{\mathbb{R}}^a, M', \partial M') \to (\xi, X, \partial X)$ where $(f', \partial f')\colon (M', \partial M') \to (X, \partial X)$ is a homotopy equivalence.*

This procedure yields an h-cobordism from N_0 to N_1, but we want to obtain a trivial bordism. In order to carry this out, we have to discuss simple bordisms. This is formalized by equipping most of our current objects with some decoration.

1.2. **Whitehead torsion and L-groups with decoration.** Recall:

$$K_1(R) = \mathrm{GL}(R)/[\mathrm{GL}(R), \mathrm{GL}(R)]$$

(1) $\widetilde{K}_1(R) = K_1(R)/\pm 1 \ni$ Whitehead torsion of a finite contractible based free R-chain complex
(2) $K_1(R)/\{\pm\gamma\} \ni$ Whitehead torsion of an h-cobordism.

We choose a subgroup U of $K_1(R)$.

Example 1.7. In the following considerations, the following cases will be important, in all of them $R = \mathbb{Z}\pi$.

(1) $U = \{\pm 1\}$
(2) $U = \{\pm\gamma\}$
(3) $U = K_1(R)$

Two bases B and B' of a free R-module V are U-equivalent if the image of the transition matrix is in U. A U-basis is an U-equivalence class of bases on V.

$\rightsquigarrow$ stable U-basis is a U-basis on $V \oplus R^a$. If $f\colon V \to W$ is an isomorphism between stably U-based R-modules, and let A be the associated matrix in a U-basis, then $[A] \in K_1(R)/U$ only depnds on f and the choice of U-bases. We then define $\tau^U(f) = [A]$.

Example 1.8. (1) In the case $U = \{\pm 1\}$, then τ^U coincides with the Whitehead torsion of stably based finite R-chain complexes defined before.
(2) $C_*(\widetilde{X})$ is (stably) U-basis for $U = \{\pm\gamma\}$. If $f : X \to Y$ is a homotopy equivalence, then $Wh(\pi_1 X) = K_1(R)/U$ and $\tau^U(f)$ coincides with the Whitehead torsion of f.

Now we always assume

$$U = \{\pm\gamma\}.$$

and we aim for a U-based surgery obstruction theorem. We have to introduce new concepts. On the algebraic side all relevant objects will be equipped with U-bases, and we finally obtain U-decorated L-groups

$$L_n^s(R) \coloneqq L_n^U(R) \text{ if } R = \mathbb{Z}\pi$$

On the topological side we replace homotopy equivalences by simple homotopy equivalences. For defining the word "simple", one has to introduce the notions of elementary expansions and elementary collapse.

$$S^{n-2} = \partial S^n_+ \subset S^{n-1} = \partial D^n \subset D^n$$

We say that Y is obtained by an elementary expansion from X if it is obtained as a pushout in the following diagram.

$$\begin{array}{ccc} S^{n-1}_+ & \longrightarrow & X \\ \big\downarrow & & \big\downarrow j \\ D^n & \longrightarrow & Y \end{array}$$

By definition this is equivalent to saying that X is obtained by an elementary collapse from Y. Strictly the expansion consists not only of the spaces X and Y, but also on the map $j : X \to Y$. Similarly one choose $p : Y \to X$ to be a homotopy inverse of j. A simple homotopy equivalence $f\colon X \to Y$ is a map such that

$$X[0] = X \longrightarrow X[1] \longrightarrow X[2] \longrightarrow \cdots \longrightarrow X[n] = Y \quad (\text{with } f\colon X[0] \to X[n])$$

exists, and each $X[i] \to X[i+1]$ is an elementary expansion or collapse.

$f\colon X \to Y$ is simple homotopy equivalence $\Leftrightarrow \tau(f) = 0$

Theorem 1.9 (s-cobordism theorem)**.** *Let (W, M_0, M_1) be an h-cobordism. It is simple $\Leftrightarrow \tau = 0 \Leftrightarrow$ it is trivial.*

Going through all statements and proofs in the previous talks, decorating many of them with stable U-bases, and using simple homotopy equivalence instead of standard homotopy equivalences, one finally obtaines a simple version of the surgery obstruction.

Theorem 1.10 (Simple Surgery Obstruction theorem for manifolds with boundary)**.** *Let $(\overline{f}, f)$ be a normal map with underlying map $(f, \partial f)\colon (M, \partial M) \to (X, \partial X)$ such that $(X, \partial X)$ is a simple Poincaré pair and such that ∂f is a simple homotopy equivalence.*

(1) *We obtain a surgery obstruction $\sigma(\overline{f}, f) \in L^s_n(\mathbb{Z}\pi, w)$ which only depends on the simple normal bordism class of $(\overline{f}, f)$.*
(2) *Suppose $n \geq 5$. $\sigma(\overline{f}, f) = 0$ in $L^s_n(\mathbb{Z}\pi, w)$ iff $(\overline{f}, f)\colon (TM \oplus \underline{\mathbb{R}}^{a}, M, \partial M) \to (\xi, X, \partial X)$ can be simplified by a finite number of surgeries to a normal map $(\overline{f}', f', \partial f)\colon (TM' \oplus \underline{\mathbb{R}}^a, M', \partial M') \to (\xi, X, \partial X')$ where $(f', \partial f')\colon (M', \partial M') \to (X, \partial X)$ is a simple homotopy equivalence.*