Manifold Atlas: supplementary material for the page Formal group laws and genera

**Proposition 3.1.** The following relation holds in  $U^2(X)$ :

(1) 
$$u +_{H} v = F_{U}(u, v) = u + v + \sum_{k \ge 1, l \ge 1} \alpha_{kl} u^{k} v^{l},$$

where the coefficients  $\alpha_{kl} \in \Omega_U^{-2(k+l-1)}$  do not depend on X. The series  $F_U(u,v)$  given by (1) is a formal group law over the ring  $\Omega_U = \Omega_U^*$ .

*Proof.* We first do calculations with the universal example  $X = \mathbb{C}P^{\infty} \times \mathbb{C}P^{\infty}$ . Then

$$U^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = \Omega^*_U[[\underline{u}, \underline{v}]],$$

where  $\underline{u}, \underline{v}$  are canonical geometric cobordisms given by the projections of  $\mathbb{C}P^{\infty} \times \mathbb{C}P^{\infty}$  onto its factors. We therefore have the following relation in  $U^2(\mathbb{C}P^{\infty} \times \mathbb{C}P^{\infty})$ :

(2) 
$$\underline{u} +_{H} \underline{v} = \sum_{k,l \ge 0} \alpha_{kl} \, \underline{u}^{k} \underline{v}^{l},$$

where  $\alpha_{kl} \in \Omega_U^{-2(k+l-1)}$ .

Now let the geometric cobordisms  $u, v \in U^2(X)$  be given by maps  $f_u, f_v \colon X \to \mathbb{C}P^\infty$  respectively. Then  $u = (f_u \times f_v)^*(\underline{u}), v = (f_u \times f_v)^*(\underline{v})$  and  $u +_{_H} v = (f_u \times f_v)^*(\underline{u} +_{_H} \underline{v})$ , where  $f_u \times f_v \colon X \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$ . Applying the  $\Omega^*_U$ -module map  $(f_u \times f_v)^*$  to (2) we obtain the required formula (1). The fact that  $F_U(u, v)$  is a formal group law follows directly from the properties of the group multiplication  $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ .

Theorem 3.2 (Buchstaber).

$$F_U(u,v) = \frac{\sum_{i,j\geq 0} [H_{ij}] u^i v^j}{\left(\sum_{r\geq 0} [\mathbb{C}P^r] u^r\right) \left(\sum_{s\geq 0} [\mathbb{C}P^s] v^s\right)},$$

where  $H_{ij}$  ( $0 \le i \le j$ ) are Milnor hypersurfaces and  $H_{ji} = H_{ij}$ .

Proof. Set  $X = \mathbb{C}P^i \times \mathbb{C}P^j$  in Proposition 3.1. Consider the Poincaré–Atiyah duality map  $D: U^2(\mathbb{C}P^i \times \mathbb{C}P^j) \to U_{2(i+j)-2}(\mathbb{C}P^i \times \mathbb{C}P^j)$  and the map  $\varepsilon: U_*(\mathbb{C}P^i \times \mathbb{C}P^j) \to U_*(pt) = \Omega^U_*$  induced by the projection  $\mathbb{C}P^i \times \mathbb{C}P^j \to pt$ . Then the composition

$$\varepsilon D \colon U^2(\mathbb{C}P^i \times \mathbb{C}P^j) \to \Omega^U_{2(i+j)-2}$$

takes geometric cobordisms to the bordism classes of the corresponding submanifolds. In particular,  $\varepsilon D(u_{H}^{+}v) = [H_{ij}]$ ,  $\varepsilon D(u^{k}v^{l}) = [\mathbb{C}P^{i-k}][\mathbb{C}P^{j-l}]$ . Applying  $\varepsilon D$  to (1) we obtain

$$[H_{ij}] = \sum_{k,\,l} \alpha_{kl} [\mathbb{C}P^{i-k}] [\mathbb{C}P^{j-l}]$$

Therefore,

$$\sum_{i,j} [H_{ij}] u^i v^j = \left(\sum_{k,l} \alpha_{kl} u^k v^l\right) \left(\sum_{i \ge k} [\mathbb{C}P^{i-k}] u^{i-k}\right) \left(\sum_{j \ge l} [\mathbb{C}P^{j-l}] v^{j-l}\right),$$

which implies the required formula.

Theorem 3.3 (Mishchenko). The logarithm of the formal group law of geometric cobordisms is given by

$$g_U(u) = u + \sum_{k \ge 1} \frac{[\mathbb{C}P^k]}{k+1} u^{k+1} \in \Omega_U \otimes \mathbb{Q}[[u]].$$

Proof. We have

$$dg_U(u) = \frac{du}{\frac{\partial F_U(u,v)}{\partial v}\Big|_{v=0}}.$$

Using the formula of Theorem 3.2 and the identity  $H_{i0} = \mathbb{C}P^{i-1}$ , we calculate

$$dg_U(u) = \frac{1 + \sum_{k>0} [\mathbb{C}P^k] u^k}{1 + \sum_{i>0} ([H_{i1}] - [\mathbb{C}P^1] [\mathbb{C}P^{i-1}]) u^i}$$

A calculation of Chern numbers shows that  $[H_{i1}] = [\mathbb{C}P^1][\mathbb{C}P^{i-1}]$ . Therefore,  $dg_U(u) = 1 + \sum_{k>0} [\mathbb{C}P^k]u^k$ , which implies the required formula.

## **Theorem 3.4** (Quillen). The formal group law $F_U$ of geometric cobordisms is universal.

Proof. Let  $\mathcal{F}$  be the universal formal group law over a ring A. Then there is a homomorphism  $r: A \to \Omega_U$  which takes  $\mathcal{F}$  to  $F_U$ . The series  $\mathcal{F}$ , viewed as a formal group law over the ring  $A \otimes \mathbb{Q}$ , has the universality properties for all formal group laws over  $\mathbb{Q}$ -algebras. Such a formal group law is determined by its logarithm, which is a series with leading term u. It follows that if we write the logarithm of  $\mathcal{F}$  as  $\sum b_k \frac{u^{k+1}}{k+1}$  then the ring  $A \otimes \mathbb{Q}$  is the polynomial ring  $\mathbb{Q}[b_1, b_2, \ldots]$ . By Theorem 3.3,  $r(b_k) = [\mathbb{C}P^k] \in \Omega_U$ . Since  $\Omega_U \otimes \mathbb{Q} \cong \mathbb{Q}[[\mathbb{C}P^1], [\mathbb{C}P^2], \ldots]$ , this implies that  $r \otimes \mathbb{Q}$  is an isomorphism.

By the Lazard Theorem the ring A does not have torsion, so r is a monomorphism. On the other hand, Theorem 3.2 implies that the image r(A) contains the bordism classes  $[H_{ij}] \in \Omega_U$ ,  $0 \le i \le j$ . Since these classes generate the whole ring  $\Omega_U$ , the map r is onto and thus an isomorphism.

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