**Fake tori (Marek Kaluba and Wojciech Politarczyk)**

# 1 Fake complex projective spaces

## 1.1 Plumbing on $D(TS^n)$

At the beginning let us review the standard construction of plumbing.

To set the necessary notation let $TS^n$ denote the tangent bundle of a sphere, and let $D(TS^n)$ be its disk bundle. Let $G = \{V, E\}$ be a connected graph and choose a vertex $v_i \in V$ in the graph take a copy of $D(TS^n)$ denoted by $D(TS^n)_i$. Then for each edge $e \in E$ beginning or ending in the vertex choose a disk $N_{i,e}$ in the 0-section sphere $S^n \subset D(TS^n)_i$ together with small closed neighbourhood

$$N_{i,e} \times D^n \cong D^n \times D^n.$$

If necessary shrink the disks chosen for all edges, to make them disjoint and perform the following identification.

If there is an edge $e \in E$ in the graph connecting vertices $v_i$ and $v_j$ then glue the chosen neighbourhoods of the points using the following map interchanging the coordinates.

$$\begin{aligned} N_{i,e} \times D^n &\longrightarrow N_{j,e} \times D^n \\ (x, y) &\longmapsto (y, x). \end{aligned}$$

As the result we obtain a manifold with boundary $(M, \partial M)$. Its homotopy type can be easily described as the wedge of spheres $S^n \vee \cdots \vee S^n \vee S^1 \vee \ldots \vee S^1$, one $S^n$ for each vertex and $S^1$ for each loop in the graph.

**Remark 1.** We have the exact sequence of pair $(M, \partial M)$:

$$0 \to H_n(\partial M) \to H_n(M) \xrightarrow{A} H_n(M) \to H_{n-1}(M) \to 0.$$

If $A$ is an isomorphism (i.e. $\det A = \pm 1$) then $H_{n-1}(\partial M) = H_n(\partial M) = 0$ and thus $\partial M$ is a homology $(2n-1)$-sphere. Observe that the matrix $A$ provides the intersection form on $M$.

To describe the intersection form we will consider two cases.

$n$ **is even:** since even dimensional sphere has double self-intersection we see that there are 2s on the diagonal of $A$. By construction the $i, j$-element is equal $\pm 1$ if and only if there is an edge connecting $v_i$ and $v_j$, and 0s elsewhere;

$n$ **is odd:** we have 0s on the diagonal and $\pm 1$ or 0 as above.

**Theorem 1.1.** If the graph $G = \{V, E\}$ is simply connected, then $\partial M$ is $(n-2)$-connected and the homology of $\partial M$ is described in terms of the intersection matrix of $M$.

**Lemma 1.2.** If the graph $G$ is simply connected, $n \geq 3$ and $\det A = \pm 1$ then the boundary is $PL$-homeomorphic to the standard sphere

$$\partial M^{2n} \cong_{PL} S^{2n-1}.$$

## 1.2 Examples

We will make use of these two examples.

**Example 1** (Kervaire Manifold)**.** For $k = 2n+1$ plumb together two copies of $D(TS^k)$, according to the graph

(this is the Dynkin diagram of the classical group $A_2$). The result is the **Kervaire manifold with boundary** $\overline{M}_A^{4n+2}$. By what we have said above, the intersection matrix is

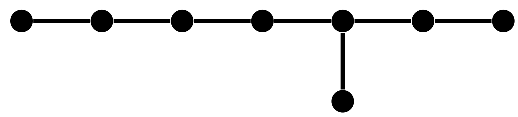
$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Therefore the boundary of $\overline{M}_A^{4n+2}$ is a $PL$-sphere by the Lemma above, hence we may attach a cone on the boundary (that is glue in a $PL$-disk $D^{4n+2}$). The result we will call the (closed) **Kervaire manifold** and denote by

$$M_A^{4n+2} \stackrel{\text{def.}}{=} \overline{M}_A^{4n+2} \cup_\partial D^{4n+2}.$$

**Remark 2.** Since we attach just a simplicial disk in the construction above we are bound to leave the smooth category. However $M_A^{2n}$ happens to be smoothable in some special cases, namely in dimensions $2, 6, 14, 30, 62$ and potentially 126, what is (of course) related to the Arf invariant 1 problem.

**Example 2.** For $k = 2n$ plumb together eight copies of $D(TS^k)$ using the graph



(this is the Dynkin diagram of the exceptional group $E_8$). The result is the **Milnor manifold with boundary** $\overline{M}_B^{4n}$. The intersection matrix is the famous $E_8$ matrix

$$B = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix},$$

which is believed to have determinant 1 (the last 2 on the diagonal corresponds to the vertex below seven others). As above we may cone of the boundary to obtain the (index 8, closed) **Milnor Manifold** which will be denoted

$$M_B^{4n} \stackrel{\text{def.}}{=} \overline{M}_B^{4n} \cup_\partial D^{4n}.$$

**Remark 3.** In the contrast to the Kervaire manifold, $M_B^{4n}$ is never smoothable. However, the $m$-fold connected sum

$$\#_m M_B^{4n} = \underbrace{M_B^{4n} \# \cdots \# M_B^{4n}}_{m \text{ times}}$$

does admit a smooth structure for certain values of $m$. This is because the boundary of $\overline{M}_B^{4n}$ generates the group $bP_{4k}$ of homotopy spheres bounding parallelizable manifolds. The smooth cases are described in the following theorem.

**Theorem 1.3** (Kervaire-Milnor, Quillen)**.** The connected sum $\#_m M_B^{4n}$ is a smooth manifold if and only if $m$ is a multiple of

$$a_n 2^{2n-2}(2^{2n-1}-1)\text{Num}\left(\frac{B_{2n}}{4n}\right),$$

which is the order of the group $bP_{4k}$.

## 1.3 The construction

**Degree 1 normal maps.**

Kervaire and Milnor closed manifolds are very important in understanding $PL$-structures on spheres. In particular, the degree 1 map $\pi\colon M_B^{4n} \to S^{4n}$ is covered by a normal bundle map and we have the associated map $S^{4n} \to G/PL$ which turns out to be a generator in $\pi_{4n}(G/PL)$. Similar result holds for the Kervaire manifold in dimensions $4n+2$.

Before we begin with the construction let us recall briefly a procedure to construct $\mathbb{C}P^{n+1}$ out of $\mathbb{C}P^n$ such that the $\mathbb{C}P^n$ embeds naturally on the first $n+1$ coordinates.

Consider $H_n \to \mathbb{C}P^n$ the Hopf bundle (or tautological complex line bundle) and its disk bundle $D(H_n)$. The boundary of the disk bundle is actually an ordinary (smooth) sphere

$$\partial D(H_n) = S(H_n) = S^{2n+1},$$

hence we may glue a cone on the boundary (that is a disk $D^{2n+2}$ along its boundary) and obtain

$$D(H_n) \cup_\partial D^{2n+2} \cong \mathbb{C}P^{n+1}.$$

In the following examples we will be trying to mimic this construction.

### 1.3.1 Fake $\mathbb{C}P^5$

Let us begin with a very graspable example of the connected sum $\mathbb{C}P^4 \#_m M_B^8$ (note: this is usually not a smooth manifold any more). There is a natural map $M_B^8 \to S^8$ (namely: the map collapsing $\overline{M}_B^8$ to a point) which induces a degree 1 normal map

$$f\colon \mathbb{C}P^4 \#_m M_B^8 \to \mathbb{C}P^4 \#_m S^8 \cong \mathbb{C}P^4.$$

We may now pull back the Hopf bundle to $\mathbb{C}P^4 \#_m M_B^8$

$$\begin{array}{ccc} D(f^*H_4) & \xrightarrow{f^*} & D(H_4) \\ \big\downarrow & & \big\downarrow p \\ \mathbb{C}P^4 \#_m M_B^8 & \xrightarrow{f} & \mathbb{C}P^4 \end{array}$$

The sphere bundle of the Hopf bundle is the true sphere, hence restriction of $f$ to boundaries is a map

$$f|_\partial \colon S(f^*H_4) \to S(H_4) = S^9.$$

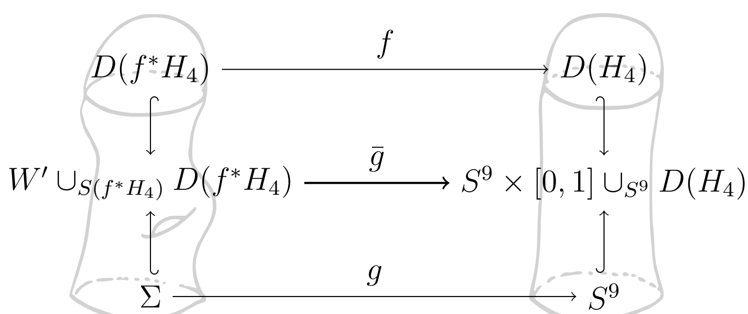
The proof of the following lemma is an exercise.

**Lemma 1.4.** The map $f|_\partial$ above is a degree 1 normal map.

Now, since we are working between simply connected odd dimensional manifolds, by surgery below the middle dimension we may assume that $f|_\partial$ is normally bordant to a homotopy equivalence $g\colon \Sigma \to S^9$. Thus by the Poincaré Conjecture $f|_\partial$ is indeed bordant to a $PL$-homeomorphism.

$$\begin{array}{ccc}
S(f^*H_4) & \xrightarrow[\text{degree 1 normal map}]{f|_\partial} & S(H_4) \\
\downarrow & & \downarrow \\
W' & \xrightarrow[\text{normal bordism}]{\bar{g}} & S^9 \times [0,1] \\
\uparrow & & \uparrow \\
\Sigma & \xrightarrow[PL\text{-homeomorphism}]{g} & S^9
\end{array}$$

Choose $W' \to S^9 \times [0,1]$ to be such bordism and glue $D(f^*H_4)$ or $D(H_4)$ on top of $W'$ or $S^9 \times [0,1]$ respectively. Now we have the following picture:




[We are abusing the notation slightly since map $\bar{g}$ in the diagram above is extended to $D(f^*H_4)$ using $f$.]

We intend to change the normal map $\bar{g}$ on the whole bordism to a homotopy equivalence. Then since $g$ is a $PL$-homeomorphism we may close the boundaries by a $PL$-disk.

**Lemma 1.5.** We can perform a surgery on the normal bordism

$$\bar{g}\colon W' \cup_{S(f^*H_4)} D(f^*H_4) \to D(H_4)$$

to make it a homotopy equivalence.

*Proof.* Proof of the lemma is just an application of the Wall realisation theorem. □

As the result we obtain a manifold $(W^{10}, \partial W)$ with boundary $PL$-homeomorphic to $S^9$ and a homotopy equivalence

$$h\colon W^{10} \to D(H_4).$$

We may now cone off the boundary of $W^{10}$ and call the result

$$\widetilde{\mathbb{C}P^5} \stackrel{\text{def.}}{=} W^{10} \cup_{S^9} D^{10}.$$

Observe that we can extend $h$ to

$$\widetilde{h}\colon \widetilde{\mathbb{C}P^5} \to \mathbb{C}P^5$$

at the same time, since the cone on the boundary is just a $PL$-disk. This is our first example of a space homotopy equivalent to the complex projective 5-space.

**Lemma 1.6.** $\widetilde{\mathbb{C}P^5}$ and $\mathbb{C}P^5$ are not homeomorphic.

*Proof.* One possible proof is to compare the $\mathcal{L}$-polynomials of these two spaces.

The above construction provides us with a homotopy equivalence, hence with a degree 1 normal map and hence with a normal invariant

$$\lambda \in [\mathbb{C}P^5, G/PL].$$

Observe that by the construction itself, $\lambda$ is trivial if and only if $m = 0$ (recall that $m$ is the multiplicity of connected sum with $M^8_B$).

The inverse image of $\mathbb{C}P^i \subset \mathbb{C}P^5$ is equal to $\mathbb{C}P^i$ for $i = 1, 2, 3$, but for $i = 4$ we have

$$f^{-1}(\mathbb{C}P^4) = \mathbb{C}P^4 \#_m M^8_B,$$

hence the surgery obstruction of $f^{-1}(\mathbb{C}P^4)$ is equal to $m$. Using some properties of the $\mathcal{L}$-polynomials we are able to compute it for our homotopy complex projective space

$$\mathcal{L}(\widetilde{\mathbb{C}P^5}) = \mathcal{L}(\mathbb{C}P^5)(1 + 8mx^4)$$

where $x$ generates the second cohomology of $\widetilde{\mathbb{C}P^5}$.

Since $\mathcal{L}$-polynomials are rational polynomials of variables $p_1, \ldots, p_n, \ldots$ (the Pontryagin classes) and the rational Pontryagin classes are homeomorphism invariants, the results follows. $\square$

### 1.3.2 Fake $\mathbb{C}P^6$

The construction above gives us a degree 1 normal map

$$\widetilde{h}\colon \widetilde{\mathbb{C}P^5} \to \mathbb{C}P^5.$$

We may pull back the canonical line bundle $H_5 \to \mathbb{C}P^5$ over $\widetilde{\mathbb{C}P^5}$.

$$\begin{array}{ccc} D(h^*H_5) & \xrightarrow{\ \widetilde{h}^*\ } & D(H_5) \\ \downarrow & & \downarrow \\ \widetilde{\mathbb{C}P^5} & \xrightarrow{\ \widetilde{h}\ } & \mathbb{C}P^5 \end{array}$$

Again by the Poincaré conjecture, the sphere bundle $S(\widetilde{h}^*(H_5)) = \partial D(\widetilde{h}^* H_5)$ is $PL$-homeomorphic to the sphere $S^{11}$. $\widetilde{\mathbb{C}P^6} = D(\widetilde{h}^* H_5) \cup_{S^{11}} D^{12}$ is a homotopy complex projective space obtained by coning off the boundary of the disk bundle.

Similarly we may form the connected sum $\widetilde{\mathbb{C}P^5} \#_n M_A^{10}$, where $M_A^{10}$ is the Kervaire manifold, and try to perform the same construction as in case of $\mathbb{C}P^4 \#_m M_B^8$. We take pull-back of $H_5$ line bundle via the natural degree 1 normal map. Now to obtain a homotopy equivalence $\partial D(\widetilde{h}^* H_5) \to \partial D(H_5)$ we have to use again a little bit of surgery.

Nevertheless, this all can be done (it is a nice exercise to check it!) and there exists a manifold $W^{12}$ such that

$$\widetilde{k}\colon D(\widetilde{h}^* H_5) \cup W^{12} \to D(H_5)$$

is a homotopy equivalence of manifolds with boundary. Finally we may extend the homotopy equivalences to the cones on boundaries using the same argument. The resulting space will be denoted by

$$\widehat{\mathbb{C}P^6} \overset{\text{def.}}{=} D(\widetilde{h}^* H_5) \cup W^{12} \cup_{S^{11}} D^{12}$$

**Remark 4.** The same is true for $\widetilde{\mathbb{C}P^5}$ replaced by $\mathbb{C}P^5$. The result will be denoted by $\overline{h}\colon \overline{\mathbb{C}P^6} \to \mathbb{C}P^6$.

**Lemma 1.7.** $\mathbb{C}P^6$, $\widetilde{\mathbb{C}P^6}$, $\widehat{\mathbb{C}P^6}$ and $\overline{\mathbb{C}P^6}$ are topologically distinct homotopy projective spaces.

The proof of the above lemma will become clear once we learn the machinery of splitting invariants which will be covered in the next section. Nevertheless we can already give sequences of the splitting invariants what will suffice as a proof after learning the classification in the next section.

These are

$$\begin{array}{ll} \mathbb{C}P^6 & (0,0,0,0) \\ \widetilde{\mathbb{C}P^6} & (0,0,m,0) \\ \overline{\mathbb{C}P^6} & (0,0,0,n \pmod 2) \\ \widehat{\mathbb{C}P^6} & (0,0,m,n \pmod 2) \end{array}$$

(the $i$-th element of the sequence is roughly speaking the surgery obstruction of the pre-image of $\mathbb{C}P^i$).

### 1.3.3 Free actions on spheres (a remark)

Observe the orbit space of the standard free action of $S^1$ on $S^{2n+1} \subset \mathbb{C}^{2n}$ is the complex projective space. The easiest example is the original Hopf fibration

$$S^1 \hookrightarrow S^3 \to S^2 = \mathbb{C}P^1.$$

There is a theorem stating that any free action of $S^1$ on the $(2n+1)$-sphere gives rise to a homotopy complex projective space $h\mathbb{C}P^n$ sitting in the obvious fibration

$$S^1 \hookrightarrow S^{2n+1} \to S^{2n+1}/S^1 = h\mathbb{C}P^n,$$

and conversely, each $h\mathbb{C}P^n$ corresponds to a unique, free $S^1$-action on $S^{2n+1}$.

We may consider the join $S^1 * S^{2n+1} \cong S^{2n+3}$. Since join is a nice quotient space of the Cartesian product $S^1 \times S^{2n+1} \times [0,1]$, from (free) actions on both spheres we have an induced (free) action on the join. In this language, the construction of $\widetilde{\mathbb{C}P^6}$ above is equivalent to the following.

Given $\widetilde{\mathbb{C}P^5}$, identify the corresponding free $S^1$-action on $S^{11}$. Take a join of the $S^{11}$ with $S^1$ with the standard action. This gives us a free $S^1$ action on $S^{13}$. The quotient space is precisely $\widetilde{\mathbb{C}P^6}$ as constructed above.

## 1.4 The classification

Consider the surgery exact sequence for $\mathbb{C}P^n$.

$$0 \longrightarrow \mathcal{S}^{PL}(\mathbb{C}P^n) \longrightarrow [\mathbb{C}P^n, G/PL] \xrightarrow{\sigma} L_{2n}(\mathbb{Z})$$

Vanishing of odd dimensional $L$ groups implies that the PL structure set is a subset of the set of normal invariants whose surgery obstruction is zero. Thus in order to identify the structure set we have to compute the set of normal invariants and the surgery obstruction map. We will do it inductively.

Consider the following cofibration sequence.

$$S^{2n-1} \longrightarrow \mathbb{C}P^{n-1} \longrightarrow \mathbb{C}P^n \longrightarrow S^{2n}$$

It yields the following exact sequence of abelian groups

$$\pi_{2n}(G/PL) \longrightarrow [\mathbb{C}P^n, G/PL] \longrightarrow [\mathbb{C}P^{n-1}, G/PL] \longrightarrow \pi_{2n-1}(G/PL).$$

The Generalized Poincaré Conjecture implies that the homotopy groups of $G/PL$ are isomorphic to $L$ groups of $\mathbb{Z}$ in dimension 5 and higher, what further implies that $\pi_{2n-1}(G/PL) = 0$. Consequently the map $[\mathbb{C}P^n, G/PL] \to [\mathbb{C}P^{n-1}, G/PL]$ induced by the inclusion is surjective. Thus it remains to identify the leftmost map of the above sequence.

If $n > 2$ the following triangle commutes.

$$\begin{array}{ccc} L_{2n}(\mathbb{Z}) \cong \pi_{2n}(G/PL) & \longrightarrow & [\mathbb{C}P^n, G/PL] \\ & \searrow^{\cong} & \downarrow \sigma \\ & & L_{2n}(\mathbb{Z}) \end{array}$$

Consequently the map $\pi_{2n}(G/PL) \to [\mathbb{C}P^n, G/PL]$ is split injective. Although for $n = 2$ the sequence does not split, there exists a commutative diagram

$$\begin{array}{ccccccc}
\pi_4(G/PL) & \longrightarrow & [\mathbb{C}P^2, G/PL] & \longrightarrow & [\mathbb{C}P^1, G/PL] & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \\
\mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/_2 & \longrightarrow & 0
\end{array}$$

The isomorphism $[\mathbb{C}P^2, G/PL] \to \mathbb{Z}$ is given by the surgery obstruction map. The generator of $L_4(\mathbb{Z})$ corresponds to the normal map $\mathbb{C}P^2\#8\overline{\mathbb{C}P}^2 \to \mathbb{C}P^2$ (overline denotes reversed orientation here).

**Definition 1.1.** Let $f\colon \mathbb{C}P^n \to G/PL$ be a map. For $1 \leq k \leq n$ we define the $2k$**-th splitting invariant** as

$$s_{2k}(f) \stackrel{\text{def.}}{=} \sigma(f|_{\mathbb{C}P^k}).$$

**Remark 5.** Consider a map $f\colon \mathbb{C}P^n \to G/PL$ and let $g\colon M^{2n} \to \mathbb{C}P^n$ be the corresponding normal map of degree 1. In order to compute the splitting invariants of $f$, deform $g$ homotopically to the map which is transverse to $\mathbb{C}P^k \subset \mathbb{C}P^n$. This yields the following degree 1 normal map

$$g|_{g^{-1}(\mathbb{C}P^k)}\colon g^{-1}(\mathbb{C}P^k) \to \mathbb{C}P^k.$$

Surgery obstruction of this map equals $s_{2k}(f)$.

**Example 3.** Consider the degree one normal map $\widetilde{h}\colon \widetilde{\mathbb{C}P^5} \to \mathbb{C}P^5$ constructed in one of the previous sections. From the construction it is evident that the transverse inverse image of $\mathbb{C}P^1$, $\mathbb{C}P^2$ and $\mathbb{C}P^3$ are $\mathbb{C}P^1$, $\mathbb{C}P^2$ and $\mathbb{C}P^3$ respectively. However the transverse inverse image of $\mathbb{C}P^4$ is the connected sum $\mathbb{C}P^4\#mM^8_B$, and the associated degree one normal map has surgery obstruction equal to $m$. Thus $s_2(\widetilde{h}) = s_4(\widetilde{h}) = s_6(\widetilde{h}) = 0$ but $s_8(\widetilde{h}) = m$.

**Example 4.** Analogously as in the previous example it is easy to compute splitting invariants of the degree one normal map $\overline{h}\colon \overline{\mathbb{C}P^6} \to \mathbb{C}P^6$. All splitting invariants of $\overline{h}$ with an exception of $s_{10}(\overline{h}) = n \pmod 2$ vanish. For the normal map $\widehat{\mathbb{C}P^6} \to \mathbb{C}P^6$ the splitting invariants are zero except for $s_8 = m$ and $s_{10} = n \pmod 2$.

**Lemma 1.8.** The map given by the formula

$$\begin{aligned} [\mathbb{C}P^n, G/PL] &\longrightarrow L_4(\mathbb{Z}) \oplus L_6(\mathbb{Z}) \oplus \cdots \oplus L_{2n}(\mathbb{Z}) \\ [f] &\longmapsto (s_4(f), s_6(f), \ldots, s_{2n}(f)) \end{aligned}$$

is bijective.

*Proof.* The proof goes by induction. To establish the base case consider the commutative diagram

$$\begin{array}{ccccccc}
\pi_4(G/PL) & \longrightarrow & [\mathbb{C}P^2, G/PL] & \longrightarrow & [\mathbb{C}P^1, G/PL] & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \\
\mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/_2 & \longrightarrow & 0
\end{array}$$

Notice that the commutativity of this diagram implies that $s_4 \equiv s_2 \pmod 2$ and consequently $[\mathbb{C}P^2, G/PL]$ is determined by the surgery obstruction. Suppose now that $n > 2$ and the thesis is proved for $n-1$. Using the following split exact sequence

$$0 \longrightarrow L_{2n}(\mathbb{Z}) \longrightarrow [\mathbb{C}P^n, G/PL] \longrightarrow [\mathbb{C}P^{n-1}, G/PL] \longrightarrow 0.$$

it is easy to obtain the desired result. □

In the lemma 1.8 we have computed the set of normal invariants and the surgery obstruction map. As was said earlier this was the only missing piece of the puzzle, hence now we can identify the structure set.

**Theorem 1.9.** The following map is a bijection.

$$\begin{aligned} \mathcal{S}^{PL}(\mathbb{C}P^n) &\longrightarrow L_4(\mathbb{Z}) \oplus L_6(\mathbb{Z}) \oplus \cdots \oplus L_{2n-2}(\mathbb{Z}) \\ [f] &\longmapsto (s_4(f), s_6(f), \ldots, s_{2n-2}(f)) \end{aligned}$$

**Corollary 1.10.** All manifolds $\mathbb{C}P^6$, $\widetilde{\mathbb{C}P^6}$, $\widehat{\mathbb{C}P^6}$, $\overline{\mathbb{C}P^6}$ are not $PL$-homeomorphic.

In order to obtain the full classification of fake complex projective spaces we need to quotient out the structure set by the action of the group $\mathcal{E}(\mathbb{C}P^n)$ of homotopy classes of homotopy self-equivalences of the $2n$-dimensional complex projective space.

**Lemma 1.11.** The group $\mathcal{E}(\mathbb{C}P^n)$ consists of two elements: the homotopy class of the identity and complex conjugation.

*Proof.* By the cellular approximation theorem there is a bijection

$$[\mathbb{C}P^n, \mathbb{C}P^n] \to H^2(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}.$$

Homotopy classes of homotopy equivalences correspond to generators $\pm 1$. One generator is represented by the homotopy class of the identity map and the other one is represented by the homotopy class of the complex conjugation. □

The group $\mathcal{E}(\mathbb{C}P^n)$ turned out to be very simple. However there still remains the problem of identifying its action on the set $[\mathbb{C}P^n, G/PL]$. Sullivan was able to compute the homotopy type of $G/PL$ and using his results it possible to check that the complex conjugation acts trivially. Thus we obtain the following classification theorem.

**Theorem 1.12** (Classification theorem)**.** The $PL$-homeomorphism type of a manifold homotopy equivalent to $\mathbb{C}P^n$, for $n > 2$, is determined completely by the set of its splitting invariants $s_4, \ldots, s_{2n-2}$.