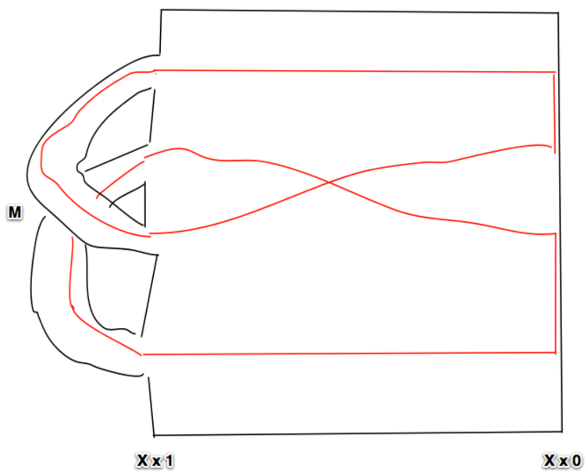
1. Manifold Atlas : Regensburg Surgery Blockseminar 2012
Wall realisation (Fabian Hebestreit)

Let X be a connected, compact smooth manifold of dimension $(n-1)$ possibly with boundary, where $n = 2k$, $k \geq 3$ and let $b \in X \setminus \partial X$ be a base point, $\pi := \pi_1(X, b)$ its fundamental group and $w\colon \pi \to \mathbb{Z}/2$ the orientation homomorphism. Furthermore let (P, λ, μ) be a $(-1)^k$-quadratic form over $(\mathbb{Z}\pi, w)$, where P is free with basis $b_1, \ldots, b_n$.

Goal. Find a manifold M of dimension n and a degree one normal map $(f, \overline{f}) : M \to X \times I$ and an isomorphism $\phi : K(\widetilde{M}, \widetilde{f}) \to P$ carrying the intersection form to λ and the selfintersection form to μ.

Idea. The plumbing manifolds arise from D^n by attaching k-handles, as was proved in Talk 13a. Therefore the idea is to obtain M by attaching k handles to one(!) side of the cylinder $X \times I$, say $X \times 1$. Note that for $X = S^{n-1}$ (or more generally $X = \partial Y$) this recovers the idea from plumbing directly, since we can always glue D^n (resp. Y) into the other side of the cylinder $(X \times 0)$: by the collaring theorem the result will be the 'same' as directly attaching handles to Y. However, to see that the manifolds constructed here really are diffeomorphic to the plumbing manifolds requires additional arguments (like Wall's classification results for $n-1$-connected n-manifolds) and is not obvious from the constructions.
The handle attachement will proceed in such a way, that the handle cores $D^k \hookrightarrow M$ will give a $\mathbb{Z}\pi$-basis for $K(\widetilde{M}, \widetilde{f})$ when completed to immersions $S^k \to M$, and these spheres will have the correct intersection and selfintersection numbers. To achieve

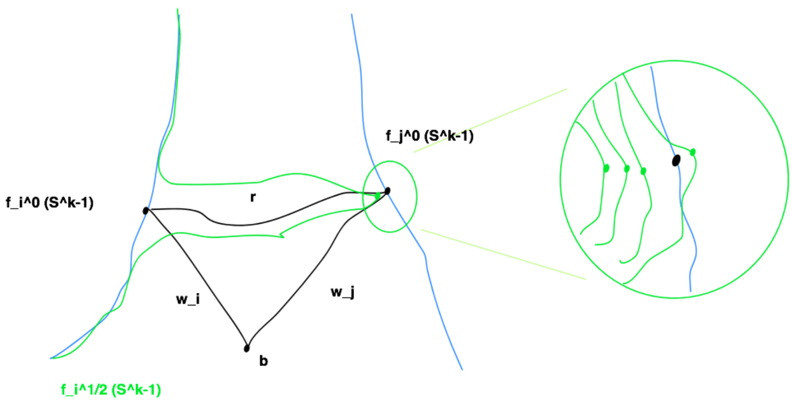


this we shall first construct the other halves $D^k \to X \times I \hookrightarrow M$ of the completions, in such a way that their boundaries will serve as attaching spheres. These halves in turn will be constructed by decomposing D^k into $S^{k-1} \times I$ and a smaller D^k glued along the boundary. On the 'small D^k'-part the maps will just consist of arbitrary trivial embeddings $f_i^0\colon D^k \hookrightarrow D^{n-1} \hookrightarrow X \times 0$ and the map $S^{k-1} \times I \to X \times I$ will arise as the 'trace' of a regular homotopy $\eta_i\colon S^{k-1} \times I \to X$ starting at $f^0_{i\,|S^{k-1}}$.

Construction. Pick r disjoint embeddings $F_i\colon D^{n-1} \to X \setminus \partial X$ and set

$$f_i^0\colon S^{k-1} \times \{0\} \subseteq S^{k-1} \times D^k \hookrightarrow D^{k-1} \xrightarrow{F_i} X \setminus \partial X$$

As indicated we shall construct $\eta_i'\colon S^{k-1}\times[0,1]\to X\times[0,1]$ with $\eta_i'(\cdot,0)=f_i^0$ and $\eta_i'(s,t)=(\eta_i(s,t),t)$ where the η_i are regular homotopies. The final η_i will arise by stacking the following:
Given $g\in\pi$. Choose some basepoint $s\in S^{k-1}$. Choose paths $w_i\colon f_i^0(s)\to b$. Choose $v\colon f_i^0(s)\to f_j^0(s)$ embedded such that $[w_j * v * w_i^{-1}]=g\in\pi$ Continue the path v a little to both sides and then take a tubular neighbourhood of it. Also pick a ball around $f_j^0(s)$. Now the regular homotopy is supposed to slide s from $f_i^0(s)$ along v until it lies inside the ball. This is facilitated by the tubular neighbourhood (which is trivial, since an intervall is contractible). Once inside the ball we slide s further through the image of f_j^0 and the dimensions match up in such a way, that this is possible with the entire resulting homotopy intersecting the image of f_j^0 exactly one time, and then in only one point (which by construction has preimage s under both maps).
Note that if we attached handles to $X\times I$ using the 'end' of the homotopy just contructed and f_j^0 (we still have to check that this is even possible), the intersection number of the completed cores would be $\pm g$; this can be seen by just tracing through the definition of intersection numbers (the sign issue is exercise ???). A similar



construction works for selfintersection numbers and we can step by step introduce the desired intersection and selfintersection numbers (which do not yet really make sense since we only have 'half' an immersed k-sphere), since our construction only ever moved one of the two given immersed $k-1$-spheres. The resulting homotopies we will call $\eta_i : S^{k-1}\times I\to X$ and we now want to attach handles to $X\times 1\subset X\times I$ via the $f_i^1 := \eta_i'(-,1)$. For this we need to construct an extension of f_i^1 to a map $S^{k-1}\times D^k\hookrightarrow X\times 1$, but this can be achieved by 'pulling back' such an extension from f_i^0 via η_i (remember f_i^0 arose as the restriction of an embedding $D^{n-1}\hookrightarrow X$ and therefore has trivial normal bundle).

$$\begin{array}{ccc} S^{k-1} & \xrightarrow{\;f_i^1\;} & X\times\{1\} \\ \updownarrow & \nearrow_{\exists} & \\ S^{k-1}\times D^k & & \end{array}$$

Call the resulting manifold M. As promised, it is a cylinder over X with k-handles attached on one side. Note now that the attaching spheres f_i^1 are nullhomotopic in

X hence also in $X \times I$ (again they are homotopic to the f_i^0's and these had images in disks). That means we can (almost) extend the identity map $X \times I \to X \times I$ to a map $f : M \to X \times I$ (mapping boundary to boundary) by only changing it in a small neighbourhood of $X \times 1$ with the image of the handles also lying in that neighbouhood (or a different one, that doesn't matter). This map f then certainly has degree one, it is a diffeomorphism on $X \times 0$ (namely the identity) and can be covered by a normal map.

$$\begin{array}{ccc} X \times I & \xrightarrow{\text{id}} & X \times I \\ \updownarrow & \nearrow_{\exists f} & \\ M & & \end{array}$$

Now to compute $K(\widetilde{M}, \widetilde{f})$ use a snakelemma type argument to show that the indicated isomorphism in the following diagram really is an isomorphism.

$$\begin{array}{ccccc} 0 & \longrightarrow & K_k(\widetilde{M}) & \xrightarrow{\cong} & K_k(\widetilde{M}, \widetilde{X \times I}) \\ \downarrow & & \downarrow & & \downarrow \\ H_k(\widetilde{X} \times I) & \longrightarrow & H_k(\widetilde{M}) & \longrightarrow & H_k(\widetilde{M}, \widetilde{X \times I}) \\ \downarrow & & \downarrow & & \downarrow \\ H_k(\widetilde{X} \times I) & \longrightarrow & H_k(\widetilde{X \times I}) & \longrightarrow & H_k(\widetilde{X \times I}, \widetilde{X \times I}) \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

(here the horizontal sequences are part of the long exact sequence of the pair $(\widetilde{M}, \widetilde{X \times I})$ and the vertical ones are kernel/cokernel sequences. Now $K(\widetilde{M}, \widetilde{X \times I})$ has bases corresponding to (lifts of) the handles. Choosing such a basis thus gives an isomorphism $K(\widetilde{M}, \widetilde{f}) \to P$ and to compute its intersection and selfintersection we may complete the cores arbitrarily and compute the geometric intersection of the resulting spheres. However piecing together a core, its corresponding lift $\tilde{\eta}_i$ and a lift of the embedded disk at the start of η_i does give such a completion and unwinding the definition of the geoemetric intersection and selfintersection numbers then gives the desired result.

Note that our construction has the following additional properties: The boundary of M is naturally decomposed into two pieces, say $\partial^0 M$ and $\partial^1 M$: the unchanged boundary of the cylinder together with $(\partial X) \times I$ (which has to be considered as a subset of M for present purposes) on the one side and the surgered boundary on the other. These pieces map to $X \times 0$ and $(X \times 1) \cup ((\partial X) \times I)$ respectively, so f really is a map of manifold triads. We can certainly arrange (and have done so given correct interpretation of the vague constructions above) for f to restrict to a diffeomorphism $\partial^0 M \to (X \times 1) \cup ((\partial X) \times I)$ and by the same argument used in the plumbing construction, the restriction of f to $\partial^1 M \to X \times 1$ will be a homotopy equivalence if and only if λ, that is the intersection form of the surgery kernel of f, is non-degenerate.

We have thus 'proved' half of (the non-simple version of)

Theorem 1.1 (Wall's realization theorem)**.** *Given a compact connected manifold X of dimension $n-1 \geq 5$ and a point $b \in X \setminus \partial X$, every class in $L_n^{h,s}(\mathbb{Z}[\pi(X,b)], w)$ (where w is the orientation character of X) can be realized as the (simple) surgery obstruction of a normal map $M \to X \times I$.*