The (smooth) surgery exact sequence (Christoph Winges)

Definition 1.1. Let X be an n-dimensional Poincaré complex, M_i closed n-manifolds, $f_i\colon M_i \to X$ simple homotopy equivalences, $i = 0, 1$.

We call f_0 and f_1 *equivalent* if there is a simple homotopy equivalence

$$(W, \partial_0 W, \partial_1 W) \xrightarrow{F} (X \times I, X \times \{0\}, X \times \{1\})$$

together with degree one diffeomorphisms $g_0\colon M_0 \to \partial_0 W$, $g_1\colon M_1^- \to \partial_1 W$ such that $\partial_i F \circ g_i = f_i$.

The collection of equivalence classes of such maps is the simple structure set $\mathscr{S}_n^s(X)$. If X is a manifold, the structure set has a canonical basepoint $*$ given by $\mathrm{id}\colon X \to X$.

Remark 1.2. (1) Note that any cobordism W as above is automatically an h-cobordism. So the s-cobordism theorem applies to show that the equivalence relation for the $\mathscr{S}_n^s(X)$ reduces to the existence of a homotopy commutative triangle

$$\begin{array}{ccc} M_0 & \xrightarrow{\cong} & M_1 \\ {\scriptstyle f_0}\downarrow & \swarrow {\scriptstyle f_1} & \\ X & & \end{array}$$

, where the horizontal map is a degree one diffeomorphism.

(2) Let $\mathscr{E}^s(X) \coloneqq \{$homotopy classes of simple homotopy equivalences$\}$. Then $\mathscr{E}^s(X)$ acts on $\mathscr{S}_n^s(X)$ via composition. The obvious forgetful map

$$\mathscr{S}_n^s(X) \to \mathscr{F}^s(X) \coloneqq \{\text{diffeomorphism classes of manifolds simply homotopy equivalent to } X\}$$

factors to give a bijection $\mathscr{E}^s(X)\backslash\mathscr{S}_n^s(X) \sim \mathscr{F}^s(X)$.

Definition 1.3. Let X be an n-dimensional, compact manifold and $(f, \overline{f})\colon M \to X$ a normal map of degree one such that ∂f is a diffeomorphism. A *nullbordism* of $(f, \overline{f})$ consists of

- a degree one map of manifold triads
$$F\colon (W, \partial_0 W, \partial_1 W) \to (X \times I, X \times \{0\}, \partial X \times I \cup X \times \{1\})$$
with $\partial_1 F$ a diffeomorphism,
- a degree one diffeomorphism $g\colon (M, \partial M) \to (\partial_0 W, \partial_0 W \cap \partial_1 W)$,
- a bundle map $\overline{F}\colon TW \oplus \underline{\mathbb{R}}^a \to \eta$ and
- an isomorphism $\xi \oplus \underline{\mathbb{R}}^a \cong \eta$

satisfying appropriate compatibility conditions. The definition of a nullbordism allows us to derive the notion of a *normal cobordism*.

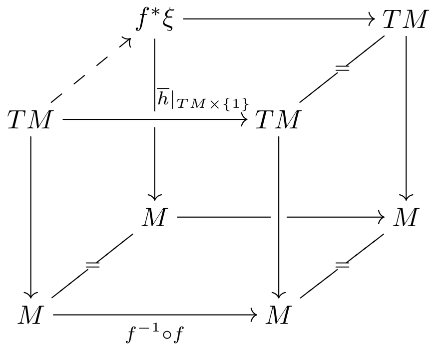
The set of normal bordism classes is denoted $\mathscr{N}_{n+1}(X, \partial X)$.

Theorem 1.4 (Browder-Novikov-Sullivan-Wall exact sequence)**.** *Let X be a smooth, connected, closed manifold, $n \geq 5$. Set $\pi \coloneqq \pi_1 X$ and let $w\colon \pi \to \{\pm 1\}$ be the orientation homomorphism. Then the following is an exact sequence of pointed sets.*

$$\mathscr{N}_{n+1}(X \times I, X \times \partial I) \xrightarrow{\quad\xi\quad} L_{n+1}(\mathbb{Z}\pi, w) \xrightarrow{\quad\partial\quad} \mathscr{S}(X) \xrightarrow{\quad\eta\quad} \mathscr{N}_n(X) \xrightarrow{\quad\sigma\quad} L_n(\mathbb{Z}\pi, w)$$

Notation. The map σ is the (simple) surgery obstruction. We give the definitions of η and ∂ for homotopy equivalences, the definitions in the simple case are analogous.

Definition of η: Let $[f\colon M \xrightarrow{\simeq} X] \in \mathscr{S}_n(X)$. Let f^{-1} be a homotopy inverse of f. Set $\xi := \left(f^{-1}\right)^* TM$. Pick a homotopy $h\colon \mathrm{id} \simeq f^{-1} \circ f$. There is a lift of h to a bundle map $\overline{h}\colon TM \times I \to TM$. The universal property of a pullback gives the dashed arrow in the following diagram:



By composing the dashed arrow with the natural map $f^*\xi \to \xi$, we obtain a map $\overline{f}\colon TM \to \xi$ and thus

$$\begin{array}{ccc} TM & \xrightarrow{\overline{f}} & \xi \quad \in \mathscr{N}^n(X) \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

Define $\eta[f] := [\overline{f}, f]$.

Definition of ∂: We define a group action $L_{n+1}(\mathbb{Z}\pi, w) \curvearrowright \mathscr{S}_n(X)$:

Let $x \in L_{n+1}(\mathbb{Z}\pi, w), [f\colon M \xrightarrow{\simeq} X] \in \mathscr{S}_n(X)$. Realize x by a normal map $F\colon (W, \partial_0 W, \partial_1 W) \to (M \times I, M \times \{0\}, M \times \{1\})$ as in the version of the Wall realization theorem given in the previous talk. Then set

$$x \cdot [f] := [f \circ \partial_1 F].$$

Define

$$\partial(x) = x \cdot [\mathrm{id}\colon X \to X].$$

Exactness.

- Exactness at $\mathscr{N}_n(X)$: This is one of the main results that were established so far.
- Exactness at $\mathscr{S}_n(X)$: Let $x \in L_{n+1}(\mathbb{Z}\pi, w)$. Realize x by a normal map $F\colon (W, \partial W) \to (X \times I, XI)$. Then $\partial_1 F$ is bordant to a diffeomorphism (as witnessed by W), so $\eta([\partial_1 F]) = *$.

  Suppose now that $\eta([f\colon M \to X]) = *$. Then $\eta([f])$ is normally bordant to a diffeomorphism. Pick any cobordism witnessing this and take its surgery obstruction to obtain a preimage of $[f]$.
- Exactness at $L_{n+1}(\mathbb{Z}\pi, w)$: Start with an element $[(f, \overline{f})] \in \mathscr{N}_{n+1}(X \times I, X \times \partial I)$. Obviously, a realization of $\sigma([(f, \overline{f})])$ is given by $(f, \overline{f})$ itself; since ∂f is a diffeomorphism, we have that $\partial(\sigma([(f, \overline{f})])) = * \in \mathscr{S}_n(X)$.

  Now let $x \in L_{n+1}(\mathbb{Z}\pi, w)$ be mapped to $*$ via ∂, i.e. if we realize x by a normal map F as in the Wall realization theorem, then $\partial_1 F$ is h-cobordant to a diffeomorphism. Pick any homotopy equivalence F' that witnesses this (cf. the definition of the structure set), then glue F and F' along the common boundary $\partial_1 F$ to get a normal map F''. Now $\partial F''$ is a diffeomorphism, so we have defined an element in $\mathscr{N}_{n+1}(X \times I, X \times \partial I)$. Since the surgery obstruction behaves additively with respect to gluing, and since an h-cobordism has surgery obstruction 0, the element $[F'']$ is a preimage of x.

Remark 1.5. The statement about exactness at $\mathscr{S}_n(X)$ can be strengthened: Two elements in the structure set have the same image under η if and only if they lie in the same $L_{n+1}(\mathbb{Z}\pi, w)$-orbit.

Example 1.6 (Homotopy spheres). Construct a map

$$\gamma \colon \mathscr{N}_{n+1}(S^n \times I, S^n \times \partial I) \to \mathscr{N}_{n+1}(S^{n+1}).$$

Let

$$[(f, \overline{f}) \colon (M, \partial_0 M, \partial_1 M) \to (S^n \times I, S^n \times \{0\}, S^n \times \{1\})] \in \mathscr{N}_{n+1}(S^n \times I, S^n \times \partial I).$$

Recall that ∂f is a diffeomorphism. This allows us to construct a closed manifold

$$N \coloneqq M \cup_{\partial f} D^{n+1} \times \{0,1\}\,.$$

Similarly, the map f extends to a map

$$f \cup_{\partial f} \mathrm{id}_{D^{n+1} \times \{0,1\}} \colon N \to S^n \times I \cup_{S^n \times \{0,1\}} D^{n+1} \times \{0,1\} \cong S^{n+1}.$$

We get a normal map

$$[(f', \overline{f}')] \colon N \to S^{n+1}] \in \mathscr{N}_{n+1}(S^{n+1})$$

Proposition 1.7.

γ is a bijection.

After checking that the surgery obstruction maps

$$\sigma \colon \mathscr{N}_{n+1}(S^n \times I, S^n \times \partial I) \to L_{n+1}(\mathbb{Z}[e]) \text{and} \sigma \colon \mathscr{N}_{n+1}(S^{n+1}) \to L_{n+1}(\mathbb{Z}[e])$$

agree under the identification provided by γ, this allows us to splice the various surgery exact sequences together to form one long exact sequence

$$\cdots \longrightarrow \mathscr{N}_{n+1}(S^{n+1}) \xrightarrow{\sigma} L_{n+1}(\mathbb{Z}[e]) \xrightarrow{\partial} \mathscr{S}_n(S^n) \xrightarrow{\eta} \mathscr{N}_n(S^n) \xrightarrow{\sigma} L_n(\mathbb{Z}[e]).$$