Referee's report on the paper
"Some 'converses' to intrinsic linking theorems"
by R. Karasev and A. Skopenkov

J. Conway and C. Gordon [CG83] and H. Sachs [Sa81] have proved (Theorem 1.1, here abbreviated as CGS-theorem) that for any PL embedding of the complete graph K_6 into R^3 the number of unordered pairs of cycles in K_6 whose images are disjoint and linked modulo 2, is odd.

The authors prove the following 'converse' (Proposition 1.2) of the above result:

A. For any odd integer n there is a PL embedding $K_6 \to R^3$ such that the image of any 3-cycle is unknotted and the linking number of any pair of disjoint cycles in K_6 is 0 except for one 'exceptional' pair, for which it is n.

The following result (Lemma 1.3) is a stronger version of [SS92, Lemma 1.4] by J. Segal and S. Spiez:

B. For any integers $0 \le l < k$ there is a k-complex F_- such that for any PL almost–embedding $f : F_- \to R^{k+l+1}$ the images $f\Sigma^k$ and $f\Sigma^l$ of certain subcomplexes $\Sigma^k \simeq S^k$ and $\Sigma^l \simeq S^l$ (that do not depend on f) are linked modulo 2.

This version got essentially already proved by A. Skopenkov and M. Tancer in [ST17] and is present also in [SK20e, Lemma 3.7]. Its proof in the submitted paper is based on Lemma 2.3 ([SS92, Lemma 1.1] and Lemma 2.4 [ST17, Lemma 6] and avoids the use of Smith index employed in [SS92].

One can observe that B is a direct consequence of [SS92, Lemma 1.4] by using general position. (The complex F_- in B is the same as the complex Q in the proof in [SS92].) Also, it seems that since [SS92, Lemma 1.4] can be extended to the case $l = k$ (at least for $k \ge 2$), B can be extended to this case as well.

The main result of the paper (Theorem 1.6) is the following 'converse' of B:

C. For any integers $1 \le l \le k$ and z there is a PL almost embedding $f : F_- \to R^{k+l+1}$ with the linking number $lk(f\Sigma^k, f\Sigma^l)$ equal to 2z+1.

The inductive base ($k = l = 1$) of this result is a consequence of A. The proof is based on an idea suggested by F. Frick. In Section 3 the authors present also an alternative older proof in which they apply Proposition 1.8, which is a consequence of the results of A. Skopenkov [Sk02]. (It is a variant of the well known Haefliger-Weber Theorem [We67].)

The authors state also the following version (Remark 1.5) of B which generalizes the CGS-theorem:

D. For any integers $0 \le l \le k$ there is a k-complex F' with subcomplexes $\Sigma^k_j \simeq S^k$ and $\Sigma^l_j \simeq S^l$, where j runs over all $(k+1)$-element subsets of the set $\{1, \dots, k+l+3\}$, which is PL embeddable into R^{k+l+1} and is such that for any PL almost–embedding $f : F' \to R^{k+l+1}$ the number of linked modulo 2 unordered pairs of the images Σ^k_j and Σ^l_j is odd.

It is claimed in the paper that D can be proved analogously as B.

The results of the paper are of interest. However C, the main result of the paper, deals only with some special polyhedra, defined in [SS92]. It would be nice to know whether this result can be extended to other classes of polyhedra, for example the one considered in D to get an extension of A. Also some other questions are left, see Remark 1.7, and to retain a full symmetry between B and C the case $k = l$ ought to be treated in B, see above. Finally, the techniques used are not that much different than those developed in earlier papers. To sum up, I am afraid that the submitted paper does not meet the very high standards of Fundamenta Mathematicae, even though it is in my opinion publishable in many good journals.

Remark: On p. 2_1 in place of $k + 1$ should stay $k + 2$.