Proposition 6.4. *The geometric cobordism in $\mathbb{C}P^i \times \mathbb{C}P^j$ corresponding to the element $x+y \in H^2(\mathbb{C}P^i \times \mathbb{C}P^j)$ is represented by the submanifold H_{ij}. In particular, the image of the fundamental class $\langle H_{ij}\rangle$ in $H_{2(i+j-1)}(\mathbb{C}P^i \times \mathbb{C}P^j)$ is Poincaré dual to $x+y$.*

Proof. We have $x + y = c_1(p_1^*(\bar\eta) \otimes p_2^*(\bar\eta))$. The classifying map $f_{x+y}\colon \mathbb{C}P^i \times \mathbb{C}P^j \to \mathbb{C}P^\infty$ is the composition of the *Segre embedding*

$$\sigma\colon \mathbb{C}P^i \times \mathbb{C}P^j \to \mathbb{C}P^{ij+i+j},$$
$$(z_0 : \ldots : z_i) \times (w_0 : \ldots : w_j) \mapsto (z_0w_0 : z_0w_1 : \ldots : z_kw_l : \ldots : z_iw_j),$$

and the embedding $\mathbb{C}P^{ij+i+j} \to \mathbb{C}P^\infty$. The codimension 2 submanifold in $\mathbb{C}P^i \times \mathbb{C}P^j$ corresponding to the cohomology class $x+y$ is obtained as the inverse image $\sigma^{-1}(H)$ of a generally positioned hyperplane in $\mathbb{C}P^{ij+i+j}$ (i.e. a hyperplane H transverse to the image of the Segre embedding). By its definition, the Milnor hypersurface is exactly $\sigma^{-1}(H)$ for one such hyperplane H. $\square$

Lemma 6.5. *We have*

$$s_{i+j-1}[H_{ij}] = \begin{cases} j, & \text{if } i = 0, \text{ i.e. } H_{ij} = \mathbb{C}P^{j-1}; \\ 2, & \text{if } i = j = 1; \\ 0, & \text{if } i = 1,\ j > 1; \\ -\binom{i+j}{i}, & \text{if } i > 1. \end{cases}$$

Proof. Let $i = 0$. Since the stably complex structure on $H_{0j} = \mathbb{C}P^{j-1}$ is determined by the isomorphism $\mathcal{T}(\mathbb{C}P^{j-1}) \oplus \mathbb{C} \cong \bar\eta \oplus \ldots \oplus \bar\eta$ (j summands) and $x = c_1(\bar\eta)$, we have

$$s_{j-1}[\mathbb{C}P^{j-1}] = jx^{j-1}\langle \mathbb{C}P^{j-1}\rangle = j.$$

Now let $i > 0$. Then

$$s_{i+j-1}(\mathcal{T}(\mathbb{C}P^i \times \mathbb{C}P^j)) = (i+1)x^{i+j-1} + (j+1)y^{i+j-1} = \begin{cases} 2x^j + (j+1)y^j, & \text{if } i = 1; \\ 0, & \text{if } i > 1. \end{cases}$$

Denote by ν the normal bundle of the embedding $\iota\colon H_{ij} \to \mathbb{C}P^i \times \mathbb{C}P^j$. Then

$$\mathcal{T}(H_{ij}) \oplus \nu = \iota^*(\mathcal{T}(\mathbb{C}P^i \times \mathbb{C}P^j)). \tag{1}$$

Since $c_1(\nu) = \iota^*(x+y)$, we obtain $s_{i+j-1}(\nu) = \iota^*(x+y)^{i+j-1}$.

Assume $i = 1$. Then by (1) and the previous Proposition,

$$s_j[H_{1j}] = s_j(\mathcal{T}(H_{1j}))\langle H_{1j}\rangle = \iota^*(2x^j + (j+1)y^j - (x+y)^j)\langle H_{1j}\rangle$$
$$= (2x^j + (j+1)y^j - (x+y)^j)(x+y)\langle \mathbb{C}P^1 \times \mathbb{C}P^j\rangle = \begin{cases} 2, & \text{if } j = 1; \\ 0, & \text{if } j > 1. \end{cases}$$

Assume now that $i > 1$. Then $s_{i+j-1}(\mathcal{T}(\mathbb{C}P^i \times \mathbb{C}P^j)) = 0$, and we obtain from (1) and the previous Proposition that

$$s_{i+j-1}[H_{ij}] = -s_{i+j-1}(\nu)\langle H_{ij}\rangle = -\iota^*(x+y)^{i+j-1}\langle H_{ij}\rangle = -(x+y)^{i+j}\langle \mathbb{C}P^i \times \mathbb{C}P^j\rangle = -\binom{i+j}{i}.$$

$\square$