

08/28, DC: Exotic spheres II

$$\dots \rightarrow \mathcal{N}(S^n \times I, \text{rel } \partial) \xrightarrow{\cong} L_{n+1}(e) \xrightarrow{\omega} \mathcal{P}(S^n) \xrightarrow{\gamma} \mathcal{N}(S^n) \xrightarrow{\sigma} L_n(e)$$

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Theorem Kervaire - Milnor

$$0 \rightarrow bP_{n+1} \rightarrow \Theta_{n+1} \rightarrow \text{cobor}(\mathbb{Z}_n) \xrightarrow{\kappa} \mathbb{Z}/2$$

Rem.: exact sequence of abelian groups

Addendum: • $\mathcal{N}(X, \text{rel } \partial) \cong [X/\partial X, G/O]$

• X closed: $\mathcal{N}(X - (\dot{D}^n \cup \dot{D}^n), \text{rel } S^{n-1} \cup S^{n-1}) \cong \mathcal{N}(X)$

\Rightarrow (*) above

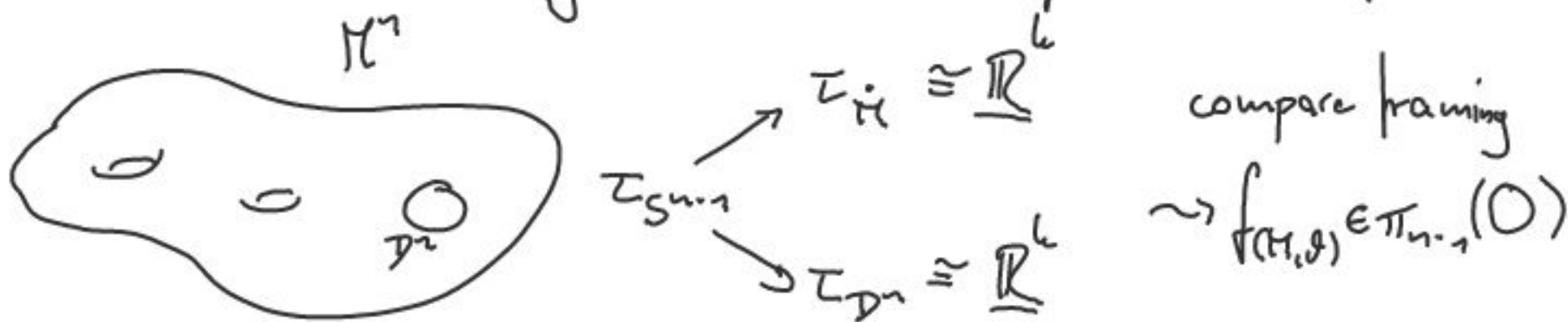
Rem.: $\pi_n(G/O) \cong \Omega_n^{\text{alm}} = \{ (M^n, \mathcal{F}) \mid \mathcal{F}: \nu_{\mathbb{R}^k} \cong \mathbb{R}^k \}$ / alm. framed bordism

[Notation: $\dot{M} \cong M$ with a small disc removed.]



Goal: $\sigma: \Omega_{4k}^{\text{alm}} \rightarrow 8\mathbb{Z}$

Find the minimal signature of an almost framed manifold.



$$\int (\mathbb{F}(f_{(M, \nu)})) = 0$$

(M, \mathcal{A}) is null bordism.

Signature Theorem [Hiizbruch]

M^{4k} closed, oriented smooth mfd.

- $\sigma(M) \in \mathbb{Z}$
- $\sigma(M) = \sigma(H^{2k}(M) \times H^{2k}(M) \rightarrow \mathbb{Q})$
- $p(M) \in H^{4*}(M; \mathbb{Z})$

Theorem

There is a polynomial $L_k \in \mathbb{Q}[p_1, p_2, \dots]$ s.t.

$$\sigma(M) = \langle L_k(p(TM)), [M] \rangle$$

e.g.: $L_1 = \frac{1}{3} p_1$, $L_2 = \frac{1}{45} (7p_2 - p_1^2)$

$$L_k = s_k p_k + \dots, \quad s_k = \frac{2^{2k} (2^{2k-1} - 1) B_k}{(2k)!} \neq 0$$

Corollary Kervaire-Milnor '63

$\Sigma^{4k} \in \Theta_{4k}$ is stably parallelisable.

Proof

$$\sigma(\Sigma^{4k}) = 0 = \langle s_k p_k(T\Sigma), [\Sigma] \rangle$$

$$\Rightarrow p_k(T\Sigma) = 0$$

$$\Rightarrow T\Sigma \cong \underline{\mathbb{R}}^k$$

$$p_k: \pi_{4k-1}(O) \rightarrow \mathbb{Z}$$

$$\pi_{4k}(G/O) \xrightarrow{\sigma} L_{4k}(e)$$

$$\sigma \left(\begin{array}{ccc} \nu_{4k} & \xrightarrow{\bar{f}} & \Sigma \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & S^{4k} \end{array} \right) = \sigma_M - \sigma_{S^{4k}}.$$

$$\sigma(f, \bar{f}) = \sigma_M = \langle -L(\nu_{4k}), [M] \rangle$$

$$= \langle \text{supp}(\nu_{4k}), [M] \rangle$$

$$\Rightarrow |bP_{4k}| = 8 \cdot a_k \cdot 2^{2k-2} \cdot \underbrace{(2^{2k-1} - 1) \text{Num}(3k/4k)}_{\text{odd}}$$

$$a_k = \begin{cases} 1 & k \text{ even} \\ 2 & k \text{ odd} \end{cases}$$

$$\rightsquigarrow bP_8 \cong \mathbb{Z}/28, \quad bP_{12} \cong \mathbb{Z}/992$$

"Law of conservation of manifolds"

$$\pi_{4k}(G/O) \rightarrow L_{4k}(e) \rightarrow \mathcal{J}(S^{4k-1})$$

112
87

$$\pi_{4k+2}(G/O) \xrightarrow{K} L_{4k+2}(e) \rightarrow \mathcal{J}(S^{4k+1})$$

112
7/2
1

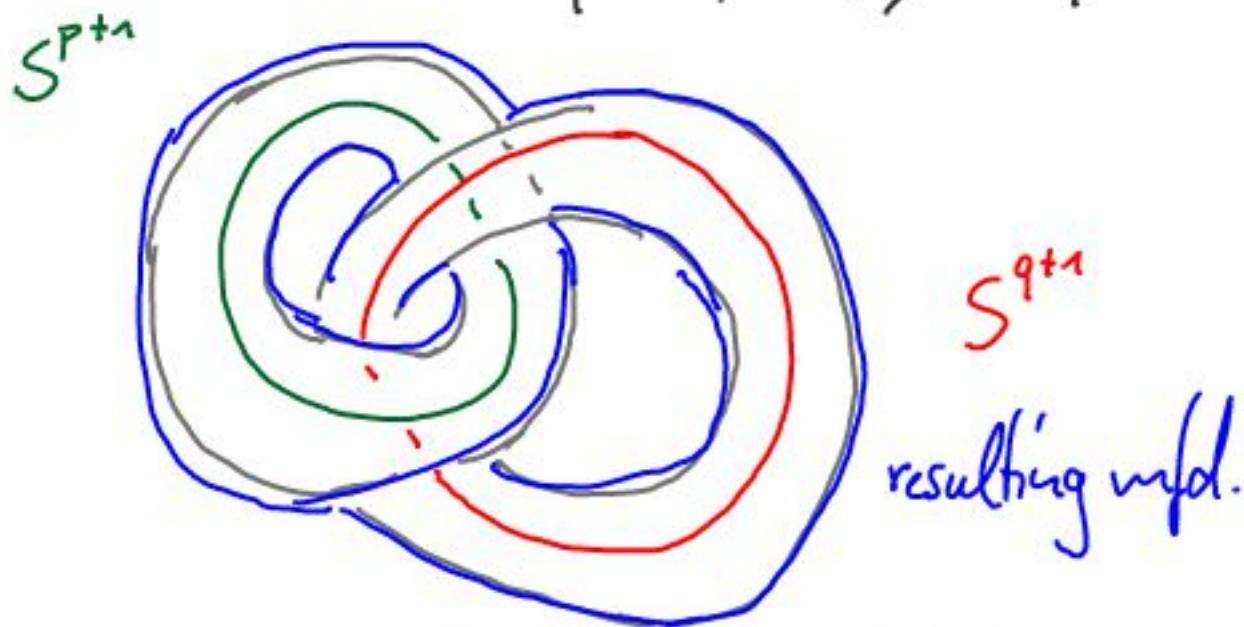
$$1 \mapsto \Sigma_k$$

see below for definition

$$\Sigma_k \cong S^{4k+1} \Leftrightarrow K_{4k+2} \neq 0$$

Construction of exotic spheres (by Plumbing)

Plumbing: $D^{p+1} \tilde{\times}_\alpha S^{q+1}$, $D^{q+1} \tilde{\times}_\beta S^{p+1}$
 $\alpha \in \pi_q(SO_{p+1})$, $\beta \in \pi_p(SO_{q+1})$



More generally, take a graph labelled by bundles. $p=q$

$$W_\mu = W^{4k}(E_8, \tau_{S^{2k}}, \dots, \tau_{S^{2k}}), \quad \partial W_\mu = \sum_{\mu} 4k-1$$

$$\lambda_{W_\mu} = E_8$$

$$bP_{4k} \cong C([\Sigma_\mu]).$$

$$\Rightarrow \sigma(\lambda_{W_\mu}) = 8$$

$$\begin{array}{c} L_{4k}(e) \\ \downarrow \quad \searrow \\ \rightarrow L_{4k}(\pi) \rightarrow \mathcal{P}(M^{4k-1}) \rightarrow \end{array}$$

$$[E_8] \cdot (N \xrightarrow{f} M) = [N \# \Sigma_\mu \rightarrow M]$$

Rem.: $W_\mu \cup D^{4k}$ is a PL manifold with no smooth structure

$$H: L_{4k+2}(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}/2 \quad (\text{Hrf invariant})$$

$$\mathcal{A} := (H, \lambda, \mu) := (\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix})$$

$$W_k := W \left(\begin{array}{ccc} & \xrightarrow{\quad} & \\ \Sigma_{5^{2k+1}} & & \Sigma_{5^{2k+1}} \end{array} \right) \quad W_k \xrightarrow{\text{deg } 1} \mathcal{D}^{4k+2}$$

$$\Sigma_{5^{2k+1}} \cong \underline{\mathbb{R}}^{2k+1} \iff 2k+1 = 1, 3, 7$$

$$(H_{W_k}, \lambda_{W_k}, \mu_{W_k}) \cong \mathcal{A}.$$

$$\partial W_k =: \Sigma_k \in \mathcal{O}_{4k+1}.$$

• Σ_k is non-standard if $4k+2 \neq 2, 6, 14, 30, 62, 126?$

Theorem Kervaire

$W_k^{10} \cup \mathcal{D}^{10}$ admits no smooth structure.

Inertia

M^n : closed, oriented smooth mfd.

$$\mathcal{O}_n \cong \mathcal{I}(M) := \{ \Sigma \mid M \# \Sigma \cong M \}, \quad \underline{\text{inertia group}}$$

$$\mathcal{I}(M) \cong \mathcal{I}_H(M) := \{ \Sigma \mid f: M \# \Sigma \cong M, f \simeq \text{id} \}$$

homotopy inertia group

Exercise: $\Sigma \in \mathcal{I}_H(M) \iff [M \# \Sigma \xrightarrow{\text{id}} M] = [\text{id}_M] \in \mathcal{J}(M).$

$$V = V_{2k+2,2} := S^{2k} \times_{\mathbb{Z}} S^{2k+1} \xrightarrow{\sim} S^{2k+1}, \quad 2\tau_{S^{2k+1}} = 0$$

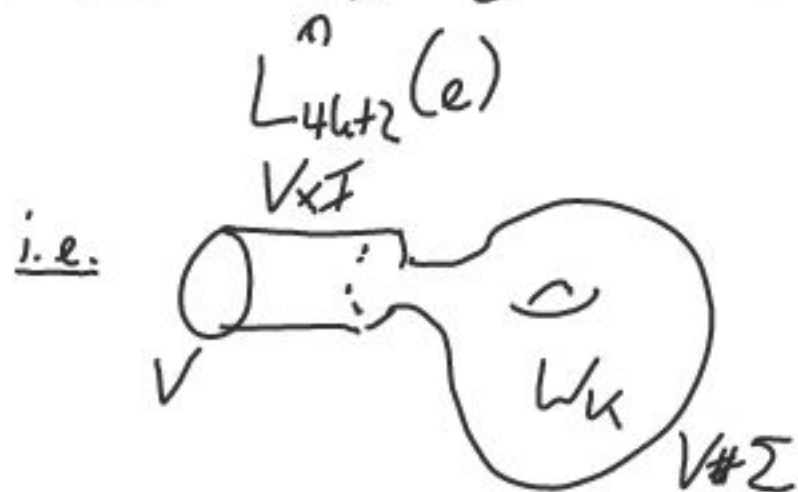
Proposition

1) $\Sigma_k \in \mathcal{I}(V)$

2) $\Sigma_k \in \mathcal{I}_H(V) \iff K_{4k+2} \neq 0$

Proof

$$V \# \Sigma = [A] \cdot [V \xrightarrow{\text{id}} V]$$



$$W := (V \times I) \cup W_k$$

$$\partial W = V \cup (V \# \Sigma_k)$$

$$H_{2k+1}(W) \cong H_{2k+1}(V) \oplus H_{2k+1}(W_k)$$

$$\cong \mathbb{Z} \oplus \mathbb{Z}^2$$

$$\cong \langle z \rangle \oplus \langle x, y \rangle$$

→ Do surgery on $z+x$.

⇒ k -cobordism from V to $V \# \Sigma$.

2) Omitted.

Exercise: Show that if $K_{4k+2} \neq 0$, then there is $f: V \xrightarrow{\sim} V$ not homotopic to a diffeomorphism.