Manifold Atlas : Regensburg Surgery Blockseminar 2012
Wall Realisation: Part (a)

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1 Plumbing

Construction (Plumbing)**.** For $i = 1, 2$ let ζ_i^q be an oriented q-plane bundle over an oriented manifold N_i and let E_i be the total space of the disk-bundle corresponding to ζ_i^q with orientation compatible with the orientations of ζ_i^q and N_i.
Let $x_i \in N_i$ and $D_i^q \times D_i^{q'} \subseteq E_i$ a neighbourhood of x_i, such that $D_i^q \times 0 \subseteq N_i$ and $y \times D_i^{q'}$ are the fibers of E_i. Let $h_\pm : D_1^q \to D_2^{q'}$ and $k_\pm : D_1^{q'} \to D_2^q$ be orientation preserving (resp. reversing) diffeomorphisms. We define the plumbing $E_1 \square E_2$ of E_1 with E_2 at x_1 and x_2 by taking $E_1 \cup E_2$ and identifying $D_1^q \times D_1^{q'}$ and $D_2^q \times D_2^{q'}$ via $I_\pm(x, y) = (k_\pm(y), h_\pm(x))$. We say that we plumb with sign $+1$ if we use I_+ and with sign -1 if we use I_-.
We get $N_i \subseteq E_i \subseteq E$, $N_1 \cap N_2 = x_1 = x_2$ is transversal and $sgn(x_1) = \pm$ depending on wether we have used I_+ or I_-.

If we choose several different points in N_1 and N_2 we can plumb E_1 and E_2 together at at those points simultaneously with prediscribed sign at each point. If we plumb E_1 and E_2 at n_{12} points with sign $+1$ we get $N_i \subseteq E_i \subseteq E_1 \square E_2$ and $i_{1*}[N_1] \cdot i_{2*}[N_2] = n_{12}$. Analogously we can plumb together several disk-bundles at once. The intersection numbers of N_i and N_j then depent on the number of points at which we plumb E_1 and E_2 together and their signs. The selfintersection numbers of the N_i depend on the Euler class $\chi(\zeta_i^q)$ in the following form:

Proposition 1.1. *Let $N^q \subseteq M^{2q}$ be a closed submanifold in the interior of M and let ζ be its normal bundle. Than the selfintersection number of N is given by the Euler number*

$$i_*[N] \cdot i_*[N] = \chi(\eta)[N]$$

Theorem 1.2. *Let $M = (m_{ij}) \in M_{n\times n}(\mathbb{Z})$ be symmetric and with even diagonal entries. Then for $k > 1$ there is a manifold W^{4k} with boundary such that:*

(1) W is $(2k-1)$-connected, ∂W is $(2k-2)$-connected, $H_{2k}(W)$ is free abelian and

(2) there is a basis of $H_{2k}(W)$ such that the matrix of intersections $H_{2k}(W) \otimes H_{2k}(W) \to \mathbb{Z}$ is given by M and

(3) there exists a normal map (f, b), $f : (W, \partial W) \to (D^{4k}, S^{4k-1})$.

In particular M is the intersection matrix on $K_{2k}(W)$.

Proof. Let $\lambda_i := \frac{m_{ii}}{2}$, $q := 2k$. Let S^q_i, $i = 1, .., n$ be q-spheres and ζ^q_i be q-plane bundles over S^q_i given by $\lambda_i \tau_{S^q} \in \pi_q(BSO_q) \cong \pi_{q-1}(SO_q)$. So the selfintersection number of S^q_i in E_i is $\chi(\zeta^q_i)[N] = \lambda_1 \chi(\tau_{S^q})[N] = 2\lambda_i = m_{ii}$ because the euler class χ is a homomorphism.
Now plumb together E_i with E_j at $|m_{ij}|$ points and with sign according to m_{ij}. For the result of the plumbing $U := E_1 \square \ldots \square E_n$ we have $S^q_i \subseteq E_i \subseteq U$ and $i_{i*}[S^q_i] \cdot i_{j*}[S^q_j] = m_{ij}$ by construction.
Since each E_i contains S^q_i as a deformation retract and the intersection between E_i and E_j is a disjoint union of disks, U has a deformation retract $\bigcup_i S^q_i$ with $S^q_i \cap S^q_j = |m_{ij}|$ points.
So U is homotopy equivalent to a wedge of q-spheres and 1-spheres.
With a Mayer-Vietoris argument, on can see, that for each component X of U the following holds:

(1) $\pi_1(\partial X) \cong \pi_1(X)$ is free and,

(2) $H_i(\partial X) = H_i(X) = 0$ for $1 < i < q$.

Now choose $S^1 \subseteq \partial X$ representing a generator g of $\pi_1(\partial X)$. We can do a surgery on S^1. Its trace V has the homotopy type of $\partial X \cup D^2$, $X_1 := X \cup_{\partial X} V$ hat the homotopy type of $X \cup D^2$ and hence $\pi_1(X_1) \cong \pi_1(X)/(g)$. Since $\dim \partial X > 3$ the same is true for $\pi_1(\partial X)$. It follows easyly from the homology sequence of the pair (X_1, X) that $H_i(X_1) \cong H_i(X)$ for $i \neq 1$.
With PD and the UCT from the exact sequence of the pair $(V, \partial X_1)$ we can conclude, that $H_i(\partial X_1) \cong H_i(\partial X)$ for $1 < i < 2q - 2$.
Doing this several times we arrive at X' with $\pi_1(X') = \pi_1(\partial X') = 0$. Let W be the boundary connected sum of those X'.
Then W is connected, $U \subseteq W$, $H_i(W) \cong H_i(U)$ for $i \neq 0, 1$ and $H_i(\partial W) \cong H_i(\partial U)$ for $1 < i < 2q - 2$. So W is $q - 1$ connected and ∂W is $q - 2$ connected. So (1) is satisfied.
In $U \subseteq W$ we have the embedded spheres S^q_i which give a basis of the homology $H_q(U) \cong H_q(W)$. M is the intersection matrix of these spheres so (2) is proven. □

Lemma 1.3. *In the construcion above ∂W is a homotopy sphere if and only if*

$$\det M \in \{\pm 1\}.$$

Proof. The map sending $H_q(W) \to \mathrm{Hom}(H_q(W), \mathbb{Z})$ given by $x \mapsto (y \mapsto x \cdot y)$, is per construction described by M, so it is an isomorphim if and only if $\det M = \pm 1$.
But this map is given as the composition

$$H_q(W) \xrightarrow{j_*} H_q(W, \partial W) \xrightarrow{PD} H^q(W) \xrightarrow{UCT} \mathrm{Hom}(H_q(W), \mathbb{Z})$$

Since $H_q(W, \partial W)$ is free ($H_q(W, \partial W) \cong H^q(W) \overset{UCT}{\cong} \mathrm{Hom}(H_q(W), \mathbb{Z})$ since $H_{q-1}(W) = 0$) this map is an isomorphism if and only if j_* is an isomorphism.
Consider the exact sequence of the pair $(W, \partial W)$.

$$0 \longrightarrow H_q(\partial W) \longrightarrow H_q(W) \xrightarrow{j_*} H_q(W, \partial W) \longrightarrow H_{q-1}(\partial W) \to 0$$

The 0 on the left hand side comes from $H_{q+1}(W, \partial W) \overset{PD}{\cong} H^{q-1}(W) \overset{UCT}{\cong} \mathrm{Hom}(H_{q-1}(W), \mathbb{Z}) = 0$, since $H_{q-2}(W), H_{q-1}(W) = 0$.
So j_* is an isomorphism if and only if $H_q(\partial W) = H_{q-1}(\partial W) = 0$, but since ∂W is a closed $(q - 2)$-connected manifold this is the case if and only if ∂W is a homotopy sphere. □

Theorem 1.4. *W is diffeomorphic to D^{2q} with q-handles attached.*

Proof. Remove the interior of a disk D^{2q} from W to get a cobordism C from S^{2q-1} to ∂W. Since ∂W is $q-1$ connected we can find a handle presentation of $C \cong S^{2q-1} \times I + \sum(i \geq q-1)^{2q} \sum(\varphi^i)$ with only handles of degree $m-1$ and higher. Looking at the dual handlebody decomposition we see that we can get rid of all handles except in degree $m-1$ and m. Since $H_{m-1}(C) = 0$ as in the proof of the s-cobordism theorem we can cancel the $m-1$ handles. This proves the theorem. □