Manifold Atlas: supplementary material for the page **Complex bordism**

Lemma. *Let $f\colon M\to N$ be a degree d map of $2i$-dimensional almost complex manifolds, and let ξ be a complex j-plane bundle over N, $j>1$. Then*
$$s_{i+j-1}[\mathbb{C}P(f^*\xi)]=d\cdot s_{i+j-1}[\mathbb{C}P(\xi)].$$

Proof. Let $p\colon \mathbb{C}P(\xi)\to N$ be the projection, γ the tautological bundle over $\mathbb{C}P(\xi)$, and $\gamma^\perp$ the complementary bundle, so that $\gamma\oplus\gamma^\perp=p^*(\xi)$. Then we have
$$\mathcal{T}(\mathbb{C}P(\xi))=p^*\mathcal{T}N\oplus\mathcal{T}_F(\mathbb{C}P(\xi)),$$
where $\mathcal{T}_F(\mathbb{C}P(\xi))$ is the tangent bundle along the fibres of the projection p. Since $\mathcal{T}_F(\mathbb{C}P(\xi))=\mathrm{Hom}(\gamma,\gamma^\perp)$ and $\mathrm{Hom}(\gamma,\gamma)=\underline{\mathbb{C}}$ (a trivial complex line bundle), we obtain
$$\mathcal{T}_F(\mathbb{C}P(\xi))\oplus\underline{\mathbb{C}}=\mathrm{Hom}(\gamma,\gamma\oplus\gamma^\perp).$$
Therefore,

(1) $$\mathcal{T}(\mathbb{C}P(\xi))\oplus\underline{\mathbb{C}}=p^*\mathcal{T}N\oplus\mathrm{Hom}(\gamma,\gamma\oplus\gamma^\perp)=p^*\mathcal{T}N\oplus\mathrm{Hom}(\gamma,p^*\xi)=p^*\mathcal{T}N\oplus(\bar\gamma\otimes p^*\xi),$$

where $\bar\gamma=\mathrm{Hom}(\gamma,\underline{\mathbb{C}})$.

The map f induces the map $F\colon\mathbb{C}P(f^*\xi)\to\mathbb{C}P(\xi)$ with the following properties:

(a) $pF=fp_1$, where $p_1\colon\mathbb{C}P(f^*\xi)\to M$ is the projection;
(b) $\deg F=\deg f$;
(c) $F^*\gamma$ is the tautological bundle over $\mathbb{C}P(f^*\xi)$.

Using (1), we obtain
$$s_{i+j-1}\big(\mathcal{T}(\mathbb{C}P(\xi))\big)=p^*s_{i+j-1}(\mathcal{T}N)+s_{i+j-1}(\bar\gamma\otimes p^*\xi)=s_{i+j-1}(\bar\gamma\otimes p^*\xi)$$
(since $i+j-1>i$), and similarly for $\mathcal{T}(\mathbb{C}P(f^*\xi))$. Thus,
$$\begin{aligned}s_{i+j-1}[\mathbb{C}P(f^*\xi)]&=s_{i+j-1}\big(\mathcal{T}(\mathbb{C}P(f^*\xi))\big)\big\langle\mathbb{C}P(f^*\xi)\big\rangle\\&=s_{i+j-1}\big((F^*\bar\gamma)\otimes p_1^*f^*\xi\big)\big\langle\mathbb{C}P(f^*\xi)\big\rangle\\&=s_{i+j-1}\big(F^*(\bar\gamma\otimes p^*\xi)\big)\big\langle\mathbb{C}P(f^*\xi)\big\rangle\\&=s_{i+j-1}(\bar\gamma\otimes p^*\xi)\langle F_*\mathbb{C}P(f^*\xi)\rangle\\&=s_{i+j-1}(\bar\gamma\otimes p^*\xi)\langle d\cdot\mathbb{C}P(\xi)\rangle\\&=d\cdot s_{i+j-1}[\mathbb{C}P(\xi)].\end{aligned}$$
□

Taras Panov
Department of Geometry and Topology
Faculty of Mathematics and Mechanics
Moscow State University, Leninskie Gory
119991 Moscow RUSSIA

E-mail address: tpanov@higeom.math.msu.su